Amplitude Analysis of the 5-Pion System
in Diffractive Pion Dissociation at COMPASS — Low $t'$

Sebastian Neubert
on behalf of the COMPASS collaboration

DPG Frühjahrstagung 2012

supported by: Maier-Leibnitz-Labor der TU und LMU München,
Cluster of Excellence: Origin and Structure of the Universe,
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Diffractive Pion Dissociation

Partial Wave Decomposition in 5-Body-Mass Bins

Resonances Embedded in the $5\pi$ Continuum

Formalism

Exploring Resonant Contributions
Diffractive Dissociation into 5 Pions

2004 COMPASS Hadron Run

- 190 GeV $\pi^-$ beam
- Pb target
- Multiplicity Trigger
- NO Recoil Detector
Diffractive Dissociation into 5 Pions

2004 COMPASS Hadron Run

- 190 GeV $\pi^-$ beam
- Pb target
- Multiplicity Trigger
- NO Recoil Detector
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins Resonances Embedded in the $5\pi$ Continuum

5\pi Data Sample
2004: $\pi + Pb \rightarrow 5\pi + Pb$

COMPASS 2004
$\pi Pb \rightarrow \pi^+\pi^-\pi^+\pi^- Pb$

Number of Events

Energy of $5\pi$ System (GeV)

COMPASS 2004
$\pi Pb \rightarrow \pi^+\pi^-\pi^+\pi^- Pb$

Number of Events

Momentum Transfer $t'$ (GeV$^2$/c$^2$)

COMPASS 2004
$\pi Pb \rightarrow \pi^+\pi^-\pi^+\pi^- Pb$

Number of Events

Events / (15 MeV/c$^2$

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
Sample for PWA

\[ t'_{\text{reco}} < 5 \times 10^{-3} \]

→ Nonexclusive background small
Isobar Model for $5\pi$ Final State

Challenges and Approaches

5-body isobar model

Isobar Decay Tree

- 11 independent variables $\tau$:
  - 4 vertices $\times$ 2 angles $+$ 3 isobar masses
- Decay amplitudes $\psi(\tau)$ in Helicity formalism
- Non-relativistic model
Isobar Model for $5\pi$ Final State

Challenges and Approaches

5-body isobar model

5-Body PWA Specials

- Decay topologies
- Many possible partial waves
- Assembly of waveset not possible by hand
- ⇒ Waveset evolution
- 284 waves tested

Isobar Decay Tree

- 11 independent variables $\tau$:
  - 4 vertices $\times$ 2 angles $+$ 3 isobar masses
- Decay amplitudes $\psi(\tau)$ in Helicity formalism
- Non-relativistic model
Evolutionary Waveset Exploration
Genetic Algorithm — 284 Waves in Pool

Control script

Amplitude pool

First generation of models

Mutator

Selector

Ranked list of surviving models

Different models

evidences

Final set of best performing models

Evidence = Goodness of fit

- Bayesian Statistics $\rightarrow$ regularized Log-Likelihood
- Takes into account model complexity
## Final Waveset
From evolutionary exploration

<table>
<thead>
<tr>
<th>$J^{PC}_{M^c}$</th>
<th>$\ell$</th>
<th>Isobar1</th>
<th>Isobar2</th>
<th>Decay Isobar2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^- + 0^+$</td>
<td>$S\ 0$</td>
<td>$\pi^- f_0(1500)$</td>
<td>$\rho(770)$</td>
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</tr>
<tr>
<td>$0^- + 0^+$</td>
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<td>$(\pi\pi)_S$</td>
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<tr>
<td>$0^- + 0^+$</td>
<td>$S\ 0$</td>
<td>$\rho(770) a_1(1260)$</td>
<td>$\pi^- (\pi\pi)_S$</td>
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</tr>
<tr>
<td>$0^- + 0^+$</td>
<td>$D\ 2$</td>
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</tr>
<tr>
<td>$0^- + 0^+$</td>
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</tr>
<tr>
<td>$1^+ + 0^+$</td>
<td>$S\ 1$</td>
<td>$\pi^- \rho(1600)$</td>
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</tr>
<tr>
<td>$1^+ + 0^+$</td>
<td>$P\ 0$</td>
<td>$\pi^- f_0(1370)$</td>
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<tr>
<td>$1^+ + 0^+$</td>
<td>$P\ 0$</td>
<td>$\pi^- (4\pi)_S (\pi\pi)_S$</td>
<td>$\pi^- (\pi\pi)_S$</td>
<td>$\rho(770)$</td>
</tr>
<tr>
<td>$1^+ + 0^+$</td>
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<tr>
<td>$1^+ + 0^+$</td>
<td>$D\ 2$</td>
<td>$(\pi\pi)_S a_2(1320)$</td>
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<td>$\rho(770)$</td>
</tr>
<tr>
<td>$1^+ + 0^+$</td>
<td>$P\ 0$</td>
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<tr>
<td>$1^+ + 0^+$</td>
<td>$S\ 1$</td>
<td>$\pi^- \eta_1(1600)$</td>
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<td>$\rho(770)$</td>
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<tr>
<td>$1^+ + 0^+$</td>
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<td>$\pi^- \rho(1700)$</td>
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</tr>
</tbody>
</table>

| $2^- + 0^+$    | $S\ 2$| $\pi^- f_2(1270)$ | $\pi^- (\pi\pi)_S$ | $a_1(1260)$ |
| $2^- + 0^+$    | $S\ 2$| $\rho(770) a_1(1260)$ | $\pi^- (\pi\pi)_S$ | $a_1(1260)$ |
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| $2^- + 0^+$    | $D\ 2$| $\pi^- f_2(1270)$ | $\pi^- (\pi\pi)_S$ | $a_1(1260)$ |
| $2^- + 0^+$    | $S\ 2$| $f_2(1270) (\pi\pi)_S$ | $\pi^- (\pi\pi)_S$ | $f_2(1270)$ |
| $2^- + 0^+$    | $P\ 1$| $\pi^- \rho(1600)$ | $\pi^- (\pi\pi)_S$ | $\rho(770)$ |
| $3^- + 0^+$    | $D\ 1$| $(\pi\pi)_S a_1(1260)$ | $\pi^- (\pi\pi)_S$ | $\rho(770)$ |
| $1^- + 0^+$    | $D\ 1$| $\rho(770) a_1(1260)$ | $\pi^- (\pi\pi)_S$ | $\rho(770)$ |

* waves used in mass independent fit.
Structures of the Spin Density Matrix
32 Waves

Re

Im
Mass Dependent Parameterization of Spin Density Matrix

\[ T_\alpha^e T_\beta^{e*} = \rho_{\alpha\beta}^e(m) = \left( \sum_k C_{\alpha k}^e A_{\alpha k}(m) \sqrt{\rho_{\alpha}(m)} \right) \left( \sum_l C_{\beta l}^e A_{\beta l}(m) \sqrt{\rho_{\beta}(m)} \right)^* \cdot \rho_{5\pi}(m) F(m) \]  

\[ (1) \]

Mass Dependent Parameterization of Spin Density Matrix

\[ T^e_\alpha T^{e*}_\beta = \rho^e_{\alpha\beta}(m) = \left(\sum_k C^e_{\alpha k} A_{\alpha k}(m) \sqrt{\rho^e_\alpha(m)}\right) \left(\sum_l C^e_{\beta l} A_{\beta l}(m) \sqrt{\rho^e_\beta(m)}\right)^* \cdot \rho_{5\pi}(m) F(m) \]  

(1)

with Breit-Wigner amplitudes:

\[ A_{\alpha k}(m, M_0, \Gamma_0) = \frac{M_0 \Gamma_0}{m^2 - M_0^2 + i\Gamma_0 M_0} \quad k = \text{resonance} \]  

(2)

and fixed width, including meson “formfactor” \( F(m) \)

In each fitted wave a coherent, constant background term is allowed, such that

\[ A_{\alpha k}(m) = c_{\alpha\text{bkg}} \quad k = \text{bkg}. \]  

(3)

The phase space factors

\[ \rho_\alpha(m) = \int |\psi^e_\alpha|^2 d\tau \]  

(4)

## Final Waveset

From evolutionary exploration

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<tr>
<th>$J^{PC}M^e$</th>
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<td>1$++0^+$</td>
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<td>1$++0^+$</td>
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<tr>
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</tr>
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</table>

2$-+0^+$ | S 2 | $\pi^- f_2(1270)$ | $\pi^+ (1^+1^-)a_1(1260)$ |
| 2$-+0^+$ | S 2 | $\rho(770)a_1(1260)$ | $\pi^- (0^+0^-)\rho(770)$ |
| 2$-+0^+$ | S 2 | $\rho(770)a_2(1320)$ | $\pi^- (2^+2^-)\rho(770)$ |
| 2$-+0^+$ | D 0 | $\rho(770)a_1(1260)$ | $\pi^- (0^+0^-)\rho(770)$ |
| 2$-+0^+$ | P 0 | $(\pi\pi)_S\pi(1800)$ | $\pi^- (0^+0^-)(\pi\pi)_S$ |
| 2$-+0^+$ | D 2 | $\pi^- f_2(1270)$ | $\pi^+ (1^+1^-)a_1(1260)$ |
| 2$-+0^+$ | S 2 | $f_2(1270)\pi(1670)$ | $\pi^- (2^+2^-)f_2(1270)$ |
| 2$-+0^+$ | P 1 | $\pi^- \rho(1600)$ | $\rho(770)(0^+0^-)(\pi\pi)_S$ |

| 3$++0^+$ | D 1 | $(\pi\pi)_S a_1(1260)$ | $\pi^- (0^+0^-)\rho(770)$ |

1$-+0^+$ | D 1 | $\rho(770)a_1(1260)$ | $\pi^- (0^+0^-)\rho(770)$ |

**FLAT**

- waves used in mass independent fit.
### 7-Resonance Fit:
Final Fit Results Overview

![Graphical Grid of Fit Results](image)

*Sebastian Neubert — Amplitude Analysis of the 5-Pion System*
The $0^{--}$ Sector
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins Resonances Embedded in the 5π Continuum

Sebastian Neubert — Amplitude Analysis of the 5-Pion System

(a) $\pi f_0(1500)$

(b) $\rho(770)a_1(1269)$

(c) Phase difference.
The $1^{++}$ Sector
Diffractive Pion Dissociation
Partial Wave Decomposition in 5-Body-Mass Bins
Resonances Embedded in the $5\pi$ Continuum

**How Many $a_1$ States Do We Need?**

Fit With TWO Resonances

(d) $\pi f_0(1370)$

(e) $\pi f_1(1285)$

(f) Phase difference.
The $2^{--}$ Sector
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins Resonances Embedded in the $5\pi$ Continuum

$2^{-+}$ Waves

(g) $\pi^- f_2(1270)$ S-wave
(h) $\rho a_1(1269)$ S-wave
(i) Phase difference
How Many $\pi_2$ States Do We Need?

Fit With ONE Resonance

![Graphs showing mass distributions and resonances](image)

Figure: Sebastian Neubert — Amplitude Analysis of the 5-Pion System 18/23
How Many $\pi_2$ States Do We Need?

Fit With TWO Resonances

![Graphs](a) (b) (c) (d)

Figure: Sebastian Neubert — Amplitude Analysis of the 5-Pion System 19/23
How Many \( \pi_2 \) States Do We Need?

Fit With THREE Resonances —

(a) COMPASS 2004 \( \pi^+\pi^-\pi^+\pi^-\pi^-\rightarrow\pi^+\pi^-\pi^+\pi^-\pi^- \)

(b) COMPASS 2004 \( \pi^+\pi^-\pi^+\pi^-\pi^-\rightarrow\pi^+\pi^-\pi^+\pi^-\pi^- \)

(c) COMPASS 2004 \( \pi^+\pi^-\pi^+\pi^-\pi^-\rightarrow\pi^+\pi^-\pi^+\pi^-\pi^- \)

(d) COMPASS 2004 \( \pi^+\pi^-\pi^+\pi^-\pi^-\rightarrow\pi^+\pi^-\pi^+\pi^-\pi^- \)
How Many $\pi_2$ States Do We Need?

Fit With FOUR Resonances

Figure:
Sebastian Neubert — Amplitude Analysis of the 5-Pion System 21/23
Final Result
Summary of Resonance Parameters

COMPASS 2004
$\pi^- \text{Pb} \rightarrow \pi^- \pi^+ \pi^- \pi^+ \pi^- \text{Pb}$

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Mass (MeV/c²)</th>
<th>Width (GeV/c²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(1300)$</td>
<td>1300</td>
<td>500</td>
</tr>
<tr>
<td>$a_1(1900)$</td>
<td>1900</td>
<td>500</td>
</tr>
<tr>
<td>$\pi_2(2100)$</td>
<td>2100</td>
<td>500</td>
</tr>
<tr>
<td>$a_1(2200)$</td>
<td>2200</td>
<td>500</td>
</tr>
<tr>
<td>$\pi_2(1670)$</td>
<td>1670</td>
<td>450</td>
</tr>
<tr>
<td>$\pi(1800)$</td>
<td>1800</td>
<td>400</td>
</tr>
<tr>
<td>$\pi_2(1880)$</td>
<td>1880</td>
<td>350</td>
</tr>
</tbody>
</table>

-preliminary-

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
Summary

- Diffractive dissociation of $\pi^-$ into $5\pi$ on lead (COMPASS 2004)
- First full 5-body PWA in $5\pi$ mass bins
- Semi-automatic model selection with genetic optimization → handle on systematic uncertainties
- First successful mass-dependent fits
  - Known states: $\pi_2(1670), \pi(1800)$ observed
  - Elusive $\pi_2(1880)$ fitted in $a_1\rho$ and $a_2\rho$
  - Fit with two $1^{++}$ resonances
  - Possible $\pi_2(2200)$ signal

Outlook

- Large data-set $\pi^- + p \rightarrow 5\pi + p$ at high $t'$ on tape
- Analysis of $4\pi$ subsystem
# Resonance Parameters

Comparison to PDG

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fit</th>
<th>PDG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resonance</td>
<td>$J^{PC}$</td>
<td>($\text{MeV}/c^2$)</td>
</tr>
<tr>
<td>$\pi(1300)$</td>
<td>$0^{-+}$</td>
<td>$1400^*$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$500^\dagger$</td>
</tr>
<tr>
<td>$\pi(1800)$</td>
<td>$0^{-+}$</td>
<td>$1781 \pm 5^{+1(+8)}_{-6(-6)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$168 \pm 9^{+5(+62)}_{-14(-15)}$</td>
</tr>
<tr>
<td>$a_1(1900)$</td>
<td>$1^{++}$</td>
<td>$1853 \pm 7^{+36(+36)}_{-6(-49)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$443 \pm 14^{+12(+98)}_{-45(-65)}$</td>
</tr>
<tr>
<td>$a_1(2200)$</td>
<td>$1^{++}$</td>
<td>$2202 \pm 8^{+15(+53)}_{-8(-11)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$402 \pm 17^{+41(+125)}_{-52(-51)}$</td>
</tr>
<tr>
<td>$\pi_2(1670)$</td>
<td>$2^{-+}$</td>
<td>$1719.0^\dagger$</td>
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<tr>
<td></td>
<td></td>
<td>$251.4^\dagger$</td>
</tr>
<tr>
<td>$\pi_2(1880)$</td>
<td>$2^{-+}$</td>
<td>$1854 \pm 6^{+6(+6)}_{-4(-9)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$259 \pm 13^{+7(+7)}_{-17(-31)}$</td>
</tr>
<tr>
<td>$\pi_2(2100)$</td>
<td>$2^{-+}$</td>
<td>$2133 \pm 12^{+7(+43)}_{-18(-18)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$448 \pm 22^{+60(+80)}_{-40(-40)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2245 \pm 60$</td>
</tr>
</tbody>
</table>

* at limit; $^\dagger$ fixed in fit

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Sebastian Neubert — Amplitude Analysis of the 5-Pion System

24/23
Isobars that have been used

<table>
<thead>
<tr>
<th>Name</th>
<th>Mass / GeV</th>
<th>$I^G J^{PC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0$</td>
<td>1370 / 1500 / 1700</td>
<td>0$^+$ (0$^{++}$)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1405</td>
<td>0$^+$ (0$^{--}$)</td>
</tr>
<tr>
<td>$\rho'$</td>
<td>1450 / 1700</td>
<td>1$^+$ (1$^{--}$)</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1235 / 1800</td>
<td>1$^+$ (1$^{+-}$)</td>
</tr>
<tr>
<td>$f_1$</td>
<td>1285 / 1420</td>
<td>0$^+$ (1$^{++}$)</td>
</tr>
<tr>
<td>$f_2$</td>
<td>1270 / 1565</td>
<td>0$^+$ (2$^{++}$)</td>
</tr>
<tr>
<td>$\eta'_2$</td>
<td>1645</td>
<td>0$^+$ (2$^{+-}$)</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>1690</td>
<td>1$^+$ (3$^{--}$)</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>1600</td>
<td>0$^+$ (1$^{--}$)</td>
</tr>
<tr>
<td>$b_0$</td>
<td>1800</td>
<td>1$^+$ (0$^{+-}$)</td>
</tr>
<tr>
<td>$b_2$</td>
<td>1800</td>
<td>2$^+$ (2$^{+-}$)</td>
</tr>
</tbody>
</table>

4$\pi$ subsystem

COMPASS 2004
$\pi^+\text{Pb} \rightarrow \pi^+\pi^-\pi^+\pi^-$

3 Entries per Event

Invariant Mass of $\pi^+\pi^-\pi^+\pi^-$ Subsystem (GeV/c$^2$)
Diffractive Pion Dissociation  Partial Wave Decomposition in 5-Body-Mass Bins  Resonances Embedded in the $\pi$ Continuum

**Isobars that have been used**

<table>
<thead>
<tr>
<th>$4\pi$ Isobars ($G = +$)</th>
<th>$3\pi$ Isobars ($G = -$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3$\pi$ subsystem</strong></td>
<td></td>
</tr>
<tr>
<td><strong>$3\pi$ subsystem</strong></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Mass / GeV</th>
<th>$I^G J^{PC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1270</td>
<td>1$^-$($1^{++}$)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1320</td>
<td>1$^-$($2^{++}$)</td>
</tr>
<tr>
<td>$\pi'$</td>
<td>1300</td>
<td>1$^-$($0^{--}$)</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>1670</td>
<td>1$^-$($2^{--}$)</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>1600</td>
<td>1$^-$($1^{+-}$)</td>
</tr>
</tbody>
</table>

---

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
Isobars that have been used

<table>
<thead>
<tr>
<th>Name</th>
<th>Mass / GeV</th>
<th>(I^G J^{PC})</th>
<th>(I^G J^{PC})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(4\pi) Isobars ((G = +))</td>
<td>(3\pi) Isobars ((G = -))</td>
<td></td>
</tr>
<tr>
<td>(f_0)</td>
<td>1370 / 1500 / 1700</td>
<td>0(^+(0^{-+}))</td>
<td></td>
</tr>
<tr>
<td>(\eta)</td>
<td>1405</td>
<td>0(^+(0^{--}))</td>
<td></td>
</tr>
<tr>
<td>(\rho')</td>
<td>1450 / 1700</td>
<td>1(^+(1^{--}))</td>
<td></td>
</tr>
<tr>
<td>(b_1)</td>
<td>1235 / 1800</td>
<td>1(^+(1^{--}))</td>
<td></td>
</tr>
<tr>
<td>(f_1)</td>
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<td>0(^+(1^{++}))</td>
<td></td>
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<tr>
<td>(f_2)</td>
<td>1270 / 1565</td>
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<td></td>
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<td>(\eta_2')</td>
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<td>(\rho_3)</td>
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<tr>
<td>(\eta_1)</td>
<td>1600</td>
<td>0(^+(1^{--}))</td>
<td></td>
</tr>
<tr>
<td>(b_0)</td>
<td>1800</td>
<td>1(^+(0^{--}))</td>
<td>(a_1)</td>
</tr>
<tr>
<td>(b_2)</td>
<td>1800</td>
<td>2(^+(2^{++}))</td>
<td></td>
</tr>
</tbody>
</table>

2\(\pi\) subsystem: \(\sigma, \rho(770), f_2(1270)\)
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins
Resonances Embedded in the $5\pi$ Continuum

Acceptance Correction

Accepted Phase-Space MC $m_{5\pi} \in [1840, 2080] \text{ MeV}/c^2$

COMPASS 2004
$\pi^+ Pb \rightarrow \pi^+\pi\pi\pi\pi Pb$
accepted phase-space MC
$m_{5\pi} \in [1840, 2080] \text{ MeV}/c^2$

Sebastian Neubert — Amplitude Analysis of the 5-Pion System 26/23
Acceptance Correction II
Accepted Phase-Space MC $m_{5\pi} \in [1840, 2080] \text{ MeV}/c^2$
Acceptance Correction III
Accepted Phase-Space MC

(a) $\cos \theta_{GJ}^{4\pi}$

(b) $\cos \theta_{GJ}^{4\pi}$

(c) $\cos \theta_{GJ}^{4\pi}$

(d) $\cos \theta_{GJ}^{4\pi}$

(e) $\cos \theta_{GJ}^{4\pi}$

(f) $\cos \theta_{GJ}^{4\pi}$

Figure:
Kinematic Validation of Fit
Data vs Weighted Monte Carlo $m_{5\pi} \in [1840, 2080]$ MeV/c²

COMPASS 2004
$\pi^+ \mathrm{Pb} \rightarrow \pi^+ \pi^+ \pi^- \pi^- \mathrm{Pb}$
Data vs weighted MC
$m_{5\pi} \in [1840, 2080]$ MeV/c²

invariant mass of $\pi\pi\pi$ system (GeV/c²)

$G\theta_1$ (cos $-0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1$)

$TY$ (cos $-3 -2 -1 0 1 2 3$

$\phi_{TV}(\pi\pi\pi^\pm)$

Invarante mass of $\pi^+\pi^+\pi^-\pi^-\pi^0$ (GeV/c²)

COMPASS 2004
$\pi^+ \mathrm{Pb} \rightarrow \pi^+ \pi^+ \pi^- \pi^- \mathrm{Pb}$
Data vs weighted MC
$m_{5\pi} \in [1840, 2080]$ MeV/c²

Sebastian Neubert — Amplitude Analysis of the 5-Pion System 29/23
Kinematic Validation of Fit
Data vs Weighted Monte Carlo $m_{5\pi} \in [1840, 2080]$ MeV/$c^2$

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins Resonances Embedded in the $5\pi$ Continuum

4$\pi$ Isospin Symmetrization
Isospin Clebsch Gordan Coefficients

The decay amplitude is isospin-symmetric, independent of $I(\pi\pi)$:

$$
\langle 1^{+} 1^{-}; 1^{+} 1^{-} | D | I_{0}^{2} \rangle = 2\sqrt{c} \langle 1^{+} 1^{-}; 1^{+} 1^{-} | D | I_{0}^{2} \rangle.
$$

For three and four pion systems this is not true:

$$
\langle \pi^{\pm} \sigma | D | 1^{\pm} 3 \rangle \quad \text{symmetric}
$$

$$
\langle \pi^{\pm} \rho_{0} | D | 1^{\pm} 3 \rangle \quad \text{antisymmetric}
$$

$$
\langle 4\pi | D | 1^{0} 4 \rangle = (1/\sqrt{2}) \langle 1^{+} 1^{-}; 1^{+} 3 | D | 1^{0} 4 \rangle \cdot (1/\sqrt{2}) \langle 1^{+} 1^{-}; 1^{+} 3 | D | 1^{0} 4 \rangle + (1/\sqrt{2}) \langle 1^{+} 1^{-}; 1^{+} 3 | D | 1^{0} 4 \rangle \cdot (1/\sqrt{2}) \langle 1^{+} 1^{-}; 1^{+} 3 | D | 1^{0} 4 \rangle.
$$

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
4π Isospin Symmetrization

Isospin Clebsch Gordan Coefficients

2π decay amplitude is isospin-symmetric, independent of $l_{(\pi\pi)}$

$$\langle 1^{\pm}; 1^{\mp} | D | l_{2}^{0} \rangle = \frac{2}{\sqrt{c}} \langle 1^{+}; 1^{-} | D | l_{2}^{0} \rangle.$$
2π decay amplitude is isospin-symmetric, independent of $I_{(\pi\pi)}$

$$\langle 1^\pm; 1^\mp | D | I_2^0 \rangle = \frac{2}{\sqrt{c}} \langle 1^+; 1^- | D | I_2^0 \rangle.$$ 

For three and four pion systems this is not true:

$$\langle \pi^\pm \sigma | D | 1_3^{\pm} \rangle \text{ symmetric}$$

$$\langle \pi^\pm \rho^0 | D | 1_3^{\pm} \rangle \text{ antisymmetric}$$
**4π Isospin Symmetrization**

**Isospin Clebsch Gordan Coefficients**

$2\pi$ decay amplitude is isospin-symmetric, independent of $I_{(\pi\pi)}$

\[
\langle 1^\pm; 1^\mp | D | I_0^2 \rangle = \frac{2}{\sqrt{c}} \langle 1^+; 1^- | D | I_0^2 \rangle.
\]

For three and four pion systems this is not true:

\[
\langle \pi^\pm \sigma | D | 1^\pm_3 \rangle \quad \text{symmetric}
\]

\[
\langle \pi^\pm \rho^0 | D | 1^\pm_3 \rangle \quad \text{antisymmetric}
\]

\[
\langle 4\pi | D | 1^0_4 \rangle = \left( \frac{1}{\sqrt{2}} \right) \langle 1^-; 1^+_3 | D | 1^0_4 \rangle \cdot \left( \frac{1}{\sqrt{2}} \right) \langle 1^+; 1^0_2 | D | 1^+_3 \rangle \cdot \frac{2}{\sqrt{2}} \langle 1^+; 1^- | D | 1^0_2 \rangle
\]

\[
+ \left( \frac{-1}{\sqrt{2}} \right) \langle 1^+_1; 1^-_3 | D | 1^0_4 \rangle \cdot \left( \frac{-1}{\sqrt{2}} \right) \langle 1^-_1; 1^0_2 | D | 1^-_3 \rangle \cdot \frac{2}{\sqrt{2}} \langle 1^+; 1^- | D | 1^0_2 \rangle
\]
4π Isospin Symmetrization

Isospin Clebsch Gordan Coefficients

2π decay amplitude is isospin-symmetric, independent of $I_{(\pi\pi)}$

$$
\langle 1^{\pm}; 1^{\mp} | D | l_2^0 \rangle = \frac{2}{\sqrt{C}} \langle 1^+; 1^- | D | l_2^0 \rangle.
$$

For three and four pion systems this is not true:

**Final Rule**

<table>
<thead>
<tr>
<th>$I(4\pi)$</th>
<th>$I(2\pi) = 1$</th>
<th>$I(2\pi) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\eta = -1$</td>
<td>$\eta = +1$</td>
</tr>
<tr>
<td>1</td>
<td>$\eta = +1$</td>
<td>$\eta = -1$</td>
</tr>
</tbody>
</table>
Exotic $4\pi$ System ... or excited $\rho$?

Correct isospin assignment essential!

$$G = (-1)^I \cdot C$$

For the $4\pi$ system $G = +$. 
Exotic $4\pi$ System ... or excited $\rho$?

Correct isospin assignment essential!

$$G = (-1)^I \cdot C$$

For the $4\pi$ system $G = +$.
Consider a $J^P = 1^-$ state

$$I = 0 \quad \Rightarrow \quad J^{PC} = 1^{-+}$$
$$I = 1 \quad \Rightarrow \quad J^{PC} = 1^{--}$$

(5)
Exotic $4\pi$ System ... or excited $\rho$?
Correct isospin assignment essential!

$$G = (-1)^I \cdot C$$

For the $4\pi$ system $G = +$.
Consider a $J^P = 1^-$ state

\[
\begin{array}{ccl}
I = 0 & \Rightarrow & J^{PC} = 1^{--} \\
I = 1 & \Rightarrow & J^{PC} = 1^{--} \\
\end{array}
\]

(5)

$$4\pi \rightarrow \pi^\pm a_1^\mp \rightarrow \pi^\pm (\pi^\mp \rho^0)$$

\[
\langle 4\pi \mid D \mid | 4^0 \rangle = \left( \frac{1}{\sqrt{2}} \right) \langle 1^-_1 ; 1^+_3 \mid D \mid 1^0_4 \rangle \cdot \left( \frac{1}{\sqrt{2}} \right) \langle 1^+_1 ; 1^0_2 \mid D \mid 1^+_3 \rangle \cdot \frac{2}{\sqrt{2}} \langle 1^+_1 ; 1^-_1 \mid D \mid 1^0_2 \rangle \\
\pm \left( \frac{-1}{\sqrt{2}} \right) \langle 1^+_1 ; 1^-_3 \mid D \mid 1^0_4 \rangle \cdot \left( \frac{1}{\sqrt{2}} \right) \langle 1^-_1 ; 1^0_2 \mid D \mid 1^-_3 \rangle \cdot \frac{2}{\sqrt{2}} \langle 1^+_1 ; 1^-_1 \mid D \mid 1^-_2 \rangle
\]
Analysis of the $4\pi$ Subsystem
Problems:

- More than one resonance in an isobar-channel (Unitarity!)
- Rescattering

Idea: (c.f. E791 $D^+ \rightarrow K^- \pi^+ \pi^+$)

- Do NOT put any model
- Replace 4-body amplitude $\rightarrow$ with piecewise constant amplitude
- Free fit of amplitude in isobar channel

Caveat:

- Need another isobar to act as interferometer
- Needs huge statistics (many fit-parameters)
**Problems:**

- More than one resonance in an isobar-channel (Unitarity!)
- Rescattering

**Idea:** (c.f. E791 $D^+ \to K^- \pi^+$)

- Do NOT put any model
- Replace 4-body amplitude
- Free fit of amplitude in isobar channel

**Caveat:**

- Need another isobar to act as interferometer
- Needs huge statistics (many fit-parameters)
$4\pi$ decay of $I^G(J^{PC}) = 0^+(1^{++})f_1$
$4\pi$ decay of $I^G(J^{PC}) = 0^+(2^{++})f_2$
Previous Search for $0^+(1^{--})$ in $\bar{p}n \rightarrow 5\pi$


- Initial state (at rest) dominated by $I^G = 1^-$, $J^{PC} = 0^{--}$ ($\bar{p}n$ s-wave)
- $4\pi$ subsystem dominated by $0^{++}$
- $\rho(1450)$ and $\rho(1700)$ found with PDG values
- Search for $\eta_1(1400)$ as Partner to $\pi_1(1400)$
  - Cannot be established (although slight increase in loglikelihood)
  - But: $0^{-+} \rightarrow \pi \eta_1$ requires P-Wave!
  - and: $\eta_1$ might be heavier while PhaseSpace is limited in $\bar{p}n$
An Interesting Amplitude
With an exotic $4\pi$ isobar

COMPASS 2004

$\pi^+\text{Pb} \rightarrow \pi^+\pi^+\pi^-\pi^-\text{Pb}$

$\Gamma(1^{++}0^+)\pi^+ \rightarrow [0^+\frac{1}{2}^+]\eta(1600) \rightarrow \pi^+ [0^+\frac{1}{2}^+]\pi^+(1269)$

Preliminary
## $\pi_2(1880)$ Mass Measurements

<table>
<thead>
<tr>
<th>Mass (MeV/c²)</th>
<th>Experiment</th>
<th>Reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1929 ± 24 ± 18</td>
<td>E852</td>
<td>$\pi^- p \rightarrow \eta\eta\pi^- p$</td>
</tr>
<tr>
<td>1876 ± 11 ± 67</td>
<td>E852</td>
<td>$\pi^- p \rightarrow \omega\pi^- \pi^0 p$</td>
</tr>
<tr>
<td>2003 ± 88 ± 148</td>
<td>E852</td>
<td>$\pi^- p \rightarrow \eta\pi^-\pi^+\pi^- p$</td>
</tr>
<tr>
<td>1880 ± 20 ± 148</td>
<td>CB</td>
<td>$\bar{p}p \rightarrow \eta\eta\pi^0\pi^0$</td>
</tr>
<tr>
<td>1836 ± 13 + 0 − 44</td>
<td>COMPASS</td>
<td>$\pi^- Pb \rightarrow \pi^-\pi^+\pi^- Pb$</td>
</tr>
<tr>
<td>1876 ± 13</td>
<td>COMPASS</td>
<td>$\pi^- Pb \rightarrow \pi^-\pi^+\pi^-\pi^+\pi^- Pb$</td>
</tr>
</tbody>
</table>

**Table:** Measured values for the mass of the $\pi_2(1880)$ resonances. As reported in [?] and compared to the COMPASS results. It is interesting that for both the $3\pi$ [?] and the new $5\pi$ result agree very well.
Diffractive Pion Dissociation Partial Wave Decomposition in 5-Body-Mass Bins Resonances Embedded in the $5\pi$ Continuum

COMPASS 2004

$\pi^+ \text{Pb} \rightarrow \pi^+ \pi^- \pi^+ \pi^- \text{Pb}$

$1'(4^{+}0')\rho(770)^0\frac{3}{2}a_2(1269)$

mass (MeV/c$^2$)

intensity

Sebastian Neubert — Amplitude Analysis of the 5-Pion System
\[
\rho_{5\pi} = a(m - m_{\text{thresh}})^5 \cdot \left[1 + b(m - m_{\text{thresh}})\right]
\]
Mass-Independent PWA

- Fit angular distributions + isobar systems in independent mass bins

\[ \sigma(\tau, m) = \sum_{\epsilon=\pm 1} \sum_{r=1}^{N_r} \left| \sum_i T_{ir}^\epsilon(m) f_i^\epsilon(t') \psi_i^\epsilon(\tau, m) \right|^2 \]

- Production amplitude
- \( t' \)-dependence (helicity “flip”)
- Decay amplitude (Helicity formalism, reflectivity basis)
Mass-Independent PWA

- Fit angular distributions + isobar systems in independent mass bins

\[ \sigma(\tau, m) = \sum_{\epsilon=\pm1} \sum_{r=1}^{N_r} \left| \sum_i T_{ir}^\epsilon(m) f_i^\epsilon(t') \psi_i^\epsilon(\tau, m) \right|^2 \]

- Production amplitude
- \( t' \)-dependence (helicity “flip”)
- Decay amplitude (Helicity formalism, reflectivity basis)

Mass-Dependent \( \chi^2 \) fit → Extract Resonance Parameters

- Parameterization of spin-density matrix elements \( \sum_r T_{ir}^\epsilon T_{jr}^{\epsilon*}(m_x) \)
- Takes into account **interference terms**
- Coherent background for some waves
Intensity distribution $I$ as a function of decay-kinematic variables $\tau$:

$$I(\tau) = \sum_{\epsilon = \pm 1} \sum_r \sum_{\alpha \in M} T^\epsilon_{\alpha r} \bar{\psi}^\epsilon_\alpha(\tau)$$

- Finite waveset $M$
- Production amplitude
- Decay amplitude
Mass Independent Amplitude Fit

Intensity distribution $\mathcal{I}$ as a function of decay-kinematic variables $\tau$:

$$
\mathcal{I}(\tau) = \sum_{\epsilon = \pm 1} \sum_r \sum_{\alpha \in M} T^{\epsilon}_\alpha r \overline{\psi}^{\epsilon}_\alpha(\tau)
$$

- Finite waveset $M$
- Production amplitude
- Decay amplitude

The likelihood $\mathcal{L}$ to observe (a specific set of) $N$ events in a bin with finite acceptance $\eta(\tau)$ (assuming a model $M$, parameters $T^{\epsilon}_{ir}$) is:

$$
P(\text{Data}| T_{ir}, M) = \mathcal{L} = \left[ \frac{N^N}{N!} e^{-N} \right] \prod_{i}^{N} \frac{\mathcal{I}(\tau_i) \eta(\tau_i) f(\tau_i)}{\int \mathcal{I}(\tau) \eta(\tau) d\rho(\tau)}
$$

with

$$
d\rho(\tau) = f(\tau) d\tau
$$

$$
\overline{N} = \int \mathcal{I}(\tau) \eta(\tau) d\rho(\tau)
$$
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\tilde{N}^N}{N!} e^{-\tilde{N}} \right] \prod_i^{N} \frac{\mathcal{I}(\tau_i)}{\tilde{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_i^{N} \mathcal{I}(\tau_i) \cdot \prod_i^{N} \eta(\tau_i) f(\tau_i) \cdot e^{-\tilde{N}} \]
Mass Independent Amplitude Fit
Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_i^N \frac{\mathcal{I}(\tau_i)}{\bar{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_i^N \mathcal{I}(\tau_i) \cdot \prod_i^N \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]

Taking the logarithm and inserting for \( \bar{N} \):

\[ \ln \mathcal{L} = -N \ln N + \sum_i^N \eta(\tau_i) f(\tau_i) + \sum_i^N \ln \mathcal{I}(\tau_i) - \int \mathcal{I}(\tau) \eta(\tau) d\rho(\tau) \]
Mass Independent Amplitude Fit
Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_{i}^{N} \frac{\mathcal{I}(\tau_i)}{\bar{N}} \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_{i}^{N} \mathcal{I}(\tau_i) \cdot \prod_{i}^{N} \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]

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drop \( -N \ln N + \sum_{i}^{N} \eta(\tau_i) f(\tau_i) \) and insert intensity parameterization

\[ \ln \mathcal{L} = \sum_{n=1}^{N_{\text{events}}} \ln \left[ \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T_{\alpha r}^{\epsilon} T_{\beta r}^{\epsilon \ast} \bar{\psi}_{\alpha}^{\epsilon}(\tau_n) \bar{\psi}_{\beta}^{\epsilon}(\tau_n)^\ast \right] - \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T_{\alpha r}^{\epsilon} T_{\beta r}^{\epsilon \ast} I_{A}^{\epsilon}_{\alpha \beta} \]
Mass Independent Amplitude Fit

Definition of LogLikelihood Function

\[ \mathcal{L} = \left[ \frac{\bar{N}^N}{N!} e^{-\bar{N}} \right] \prod_{i} I(\tau_i) \eta(\tau_i) f(\tau_i) = \frac{1}{N!} \prod_{i} I(\tau_i) \cdot \prod_{i} \eta(\tau_i) f(\tau_i) \cdot e^{-\bar{N}} \]

Taking the logarithm and inserting for \( \bar{N} \):

\[ \ln \mathcal{L} = -N \ln N + \sum_{i}^N \eta(\tau_i) f(\tau_i) + \sum_{i} \ln I(\tau_i) - \int I(\tau)\eta(\tau)d\rho(\tau) \]

drop \( -N \ln N + \sum_{i}^N \eta(\tau_i) f(\tau_i) \) and insert intensity parameterization

\[ \ln \mathcal{L} = \sum_{n=1}^{N_{\text{events}}} \ln \left[ \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T_{\alpha r}^{\epsilon} T_{\beta r}^{\epsilon*} \bar{\psi}_{\epsilon}^{\alpha}(\tau_n) \bar{\psi}_{\epsilon}^{\beta}(\tau_n)^* \right] - \sum_{\epsilon, r} \sum_{\alpha, \beta \in M} T_{\alpha r}^{\epsilon} T_{\beta r}^{\epsilon*} IA_{\alpha \beta}^{\epsilon} \]

\[ IA_{\alpha \beta}^{\epsilon} = \int \bar{\psi}_{\epsilon}^{\alpha}(\tau_n) \bar{\psi}_{\epsilon}^{\beta}(\tau_n)^* \eta(\tau)d\tau \]
Which waves to include into the waveset?
Which waves to include into the waveset?

Avoid overfitting
Which waves to include into the waveset?

Avoid overfitting

→ Data driven method
Bayes’ Theorem (for the Model Probability after Observation)

\[ P(M_k|\text{Data}) = \frac{P(\text{Data}|M_k)P(M_k)}{\sum_{k'} P(\text{Data}|M_{k'})P(M_{k'})} \]

with model-priors \( P(M_k) \) \( \sum_{k'} P(M_{k'}) = 1 \)
### How to Measure the Goodness of a Model

#### Marginal Likelihood Definition

**Bayes’ Theorem (for the Model Probability after Observation)**

\[
P(M_k|\text{Data}) = \frac{P(\text{Data}|M_k)P(M_k)}{\sum_{k'} P(\text{Data}|M_{k'})P(M_{k'})}
\]

with model-priors \(P(M_k)\) \(\sum_{k'} P(M_{k'}) = 1\)

**Marginal Likelihood or Evidence**

\[
P(\text{Data}|M_k) = \int P(\text{Data}|T^k, M_k) \underbrace{P(T^k|M_k)}_{\text{Prior}} dT^k
\]

\(P(T^k|M_k)\) contains any pre-knowledge on the model-parameters \(T\)

- Marginalization (= \(\int dT\)) is not trivial in high-dimensional spaces
- Numerically stable is only the LogLikelihood
The Occam Factor Approximation


\[ P(\text{Data}|M_k) = \int P(\text{Data}|T_k^k, M_k) P(T_k^k|M_k) \, dT_k^k \]
The Occam Factor Approximation


\[
P(\text{Data}|M_k) = \int_{\mathcal{L}} P(\text{Data}|T^k, M_k) \cdot P(T^k|M_k) \, dT^k
\]

Approximate with Laplace’s method:

\[
P(\text{Data}|M_k) \approx P(\text{Data}|T_{ML}^k, M_k) \cdot P(T_{ML}^k|M_k) \cdot \sqrt{(2\pi)^d|C_T|_{\text{Data}}}
\]

\(\text{Occam factor}\)
The Occam Factor Approximation

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- \( P(\text{Data}|T_{ML}^k, M_k) \) LogLikelihood at maximum likelihood solution \( T_{ML} \)
- \( |C_T|_{\text{Data}} \) determinant of covariance matrix
- Dimension of parameter space: \( d \)
The Occam Factor Approximation


\[ P(\text{Data}|M_k) = \int P(\text{Data}|T^k, M_k) \frac{P(T^k|M_k)}{\mathcal{L}} \, dT^k \]

Approximate with Laplace’s method:

\[ P(\text{Data}|M_k) \approx P(\text{Data}|T^k_{\text{ML}}, M_k) \cdot \frac{P(T^k_{\text{ML}}|M_k)}{\sqrt{(2\pi)^d|\mathbf{C}_T|_{\text{Data}}}} \]

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- \( |\mathbf{C}_T|_{\text{Data}} \) determinant of covariance matrix
- Dimension of parameter space: \( d \)

Logarithmic evidence:

\[ \ln P(\text{Data}|M_k) \approx \ln P(\text{Data}|T^k_{\text{ML}}, M_k) + \ln P(T^k|M_k) + \ln \sqrt{(2\pi)^d|\mathbf{C}_T|_D} \]
Final Definition

Log-Evidence

\[
\ln P(Data|M_k) \approx \ln \mathcal{L}_{ML} + \ln \sqrt{(2\pi)^d |C_T|_{Data}} - \ln V^k_T + \sum_{i \in M} \ln S_i
\]

where \( V^k_T \) is the (prior) volume of parameter space

- Models (=wavesets) compared through the Bayes-Factor
  \[
  B_{12} = \frac{P(Data|M_1)}{P(Data|M_2)}
  \]

- Interpretation according to Kass&Raftery:

<table>
<thead>
<tr>
<th>( 2 \ln B_{12} )</th>
<th>( B_{12} )</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 2</td>
<td>1 to 3</td>
<td>Not worth mentioning</td>
</tr>
<tr>
<td>2 to 6</td>
<td>3 to 20</td>
<td>Positive</td>
</tr>
<tr>
<td>6 to 10</td>
<td>20 to 150</td>
<td>Strong</td>
</tr>
<tr>
<td>&gt; 10</td>
<td>&gt; 150</td>
<td>Very strong</td>
</tr>
</tbody>
</table>

Automatic Waveset Exploration

Genetic Algorithm

Figure of Merrit

- Bayesian Statistics → regularized Log-Likelihood
- Takes into account model complexity
Automatic Waveset Exploration
Genetic Algorithm – 100 generations, population size 50

- Pool: $\sim 300$ waves
- Small wave suppression $5\sigma$
- Waveset size optimizes around 34 waves