Recent Status of the Dual Parametrization for GPDs

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Outlook

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   - Calculation of the amplitude & Abel transform tomography
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   - Why do we need analyticity in conformal spin?
   - Is it really necessary?
   - Possible way to check it.
5. Conclusions

K. S., Eur. Phys. J. A **36**, 303 (2008) [arXiv:0803.2218 [hep-ph]

Maxim V. Polyakov, K. S.,Eur. Phys. J. A **40**, 181 (2009) [arXiv:0811.2901[hep-ph]]

K. S., Eur. Phys. J. A **45**, 217,2 (2010). [arXiv:1001.2711[hep-ph]]

- The dedicated experiments provide an increasing amount of experimental data on hard exclusive processes described within GPD formalism. The extraction of GPDs from the experimental data is highly demanded.
- GPDs are complicated functions of x, ξ, and t as well as of factorization scale. The direct extraction of GPDs from the observables is an extremely difficult task. GPDs always enter the observables being integrated over x with weighting functions.
- In order to extract GPDs from the data, one usually relies on different phenomenologically motivated parameterizations and simultaneous fitting procedures for several observables. This require suitable Anzätze and theoretical development.

GPD modelling is usually guided by

1. Relation to PDFs and form factors
2. Lorentz symmetry requirements such as polynomiality property
3. Evolution properties of GPDs
4. Analyticity requirements
5. Regge phenomenology considerations
6. The insight obtained from GPD calculations in different dynamical model
7. Experimental constrains (*e.g.* skewness effect for small x_{Bj})

Conformal PW expansion for GPDs I

Main advantage of expansion of GPDs in conformal PW: trivial solution of the LO evolution equations.

- Conformal moments of quark GPDs are defined with respect to $c_n(x,\xi) = N_n \times \xi^n C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)$; Normalization: $\lim_{\xi\to 0} c_n(x,\xi) = x^n$.

$$m_n(\xi,t) = \int_{-1}^{1} dx\, c_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right) H(x,\xi,t)\,.$$

- $c_n(x,\xi)$ form a complete basis in $[-\xi,\ \xi]$ with the weight $\left(1-\frac{x^2}{\xi^2}\right)$.
- $p_n(x,\xi)$ include the weight and θ to ensure the support:

$$p_n(x,\xi) = \xi^{-n-1}\theta\left(1-\frac{x^2}{\xi^2}\right)\left(1-\frac{x^2}{\xi^2}\right) N_n^{-1}\frac{(n+1)(n+2)}{2n+3} C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)\,.$$

- Orthogonality of the basis: $$\int_{-1}^{1} dx\, p_n(x,\xi) c_n(x,\xi) = \delta_{mn}$$

Conformal PW expansion for GPDs II

Conformal PW expansion for GPDs:

$$H(x,\xi,t) = \sum_{n=0}^{\infty} p_n(x,\xi) m_n(\xi,t)\,.$$

- Conformal moments are reproduced by this series.
- Restricted support property $\not\Rightarrow$ GPD vanishes in the outer region.
- The expansion is to be understand as an ill-defined sum of generalized functions.
- Allows to factorize x, ξ and t dependence of GPDs.

Different ways to assign meaning to conformal PW expansion

1. Sommerfeld-Watson transform + Mellin-Barnes integral techniques D. Müller and A. Schäfer'05; A. Manashov, M. Kirch and A. Schafer'05;
2. - Shuvaev transform A. Shuvaev'99, J. Noritzsch'00;
   - Dual parametrization of GPDs M. Polyakov and A. Shuvaev'02;

- Sommerfeld-Watson transform:

$$H(x,\xi,t) = \frac{1}{2i}\oint_{(0)}^{(\infty)} dj\, \frac{(-1)^j}{\sin \pi j}\, p_j(x,\xi)\, m_j(\xi,t)\,.$$

- Residue theorem leads to conformal P.W. expansion $(\mathrm{Res}_{j=n}\frac{1}{\sin \pi j} = \frac{(-1)^j}{\pi})$.
- The main difficulty is to find the appropriate analytic continuation of $p_j(x,\xi)$ and $m_j(x,\xi)$ in j.
- For $p_j(x,\xi)$ the problem is solved by the so-called Schläfli integral:

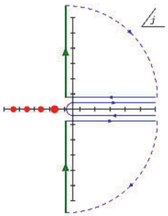
$$p_j(x,\xi) = \frac{\Gamma\left(\frac{5}{2}+j\right)(-1)^{j+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma(2+j)}\,\frac{1}{2\pi i}\oint_{-1-\epsilon}^{1+\epsilon} du\,\frac{(u^2-1)^{j+1}}{(x+u\xi)^{j+1}}\,.$$

  For integer $j=n$ and $|x|\leq\xi$ by residue theorem it gives Gegenbauer polynomials via Rodriguez formula.

Mellin-Barnes techniques in simple words II

- Important case $\xi = 0 \Rightarrow$ integral kernel for the inverse Mellin transform:

$$p_j(x,0) = x^{-j-1} \frac{\Gamma\left(\frac{5}{2}+j\right)(-1)^{j+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma(2+j)} \int_{-1}^{1} du(1-u^2)^{j+1} = \frac{(-1)^j \sin(\pi(j+1))}{\pi} \frac{1}{x^{j+1}}\,.$$



- In general $p_j(x,\xi)$ is expressed through ${}_2F_1$ hypergeometric function. Asymptotic behavior of $p_j(x,\xi)$ for $j \to \infty$ is known.
- Asymptotic behavior of m_j -?
- Integral over the large arc must vanish.

- Mellin-Barnes integral representation for GPDs:

$$H(x,\xi,t) = \frac{i}{2} \int_{c-i\infty}^{c+i\infty} dj \, \frac{(-1)^j}{\sin \pi j} p_j(x,\xi) m_j(\xi,t)\,.$$

- Simple expression for the elementary amplitude.
- Easy to include perturbative corrections.

- How to restore $f(x)$ from its Mellin moments $M_n = \int dx x^n f(x)$?

- Formal solution:
$$f(x) = \sum_{n=0}^{\infty} M_n \delta^{(n)}(x) \frac{(-1)^n}{n!}\,.$$

✓ A trick: $\delta^{(n)}(x) = \frac{1}{2\pi i}(-1)^n n! \left[\frac{1}{(x-i\epsilon)^{n+1}} - \frac{1}{(x+i\epsilon)^{n+1}}\right]$.

Define $F(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}$; then $f(x) = \frac{1}{2\pi i}\left[F(x-i\epsilon) - F(x+i\epsilon)\right]$.

Shuvaev transform:

- Introduce $f_\xi(y)$ whose Mellin moments generate Gegenbauer moments of GPD:

$$\int_0^1 dy y^n f_\xi(y) = m_n(\xi)$$

- One can explicitly construct the kernel $K(x,\xi;\,y)$ such that

$$H(x,\xi) = \int_0^1 dy\, K(x,\xi;\,y)\, f_\xi(y)\,.$$

Dual Parametrization (M. Polyakov, A. Shuvaev'02):

- Mellin moments expanded in a set of suitable orthogonal polynomials. E.g. partial waves of the t-channel (t-channel refers to $\bar{h}h \to \gamma^*\gamma$):

$$N_n^{-1}\frac{(n+1)(n+2)}{2n+3}m_n(\xi,t) = \xi^{n+1}\sum_{l=0}^{n+1} B_{nl}(t)P_l\left(\frac{1}{\xi}\right)$$

Conformal PW expansion id then rewritten as:

$$H(x,\xi,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1} B_{nl}(t)\,\theta\left(1-\frac{x^2}{\xi^2}\right)\left(1-\frac{x^2}{\xi^2}\right)C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)P_l\left(\frac{1}{\xi}\right)$$

- Introduce $Q_k(y,t)$ that generate the generalized F.F.:

$$B_{n\ n+1-k}(t) = \int_0^1 dy y^n Q_k(y,t)\,.$$

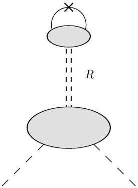
- GPD is given by the convolution with the set of kernels:

$$H(x,\xi,t) = \sum_{k=0}^{\infty}\int_0^1 dy K^{(k)}(x,\xi,y)Q_k(y,t)\,.$$

- Conformal PW expansion converges for $\xi > 1$.
- By means of the crossing relation one gets conformal PW expansion for two particle GDAs.

$$\frac{x}{\xi} \leftrightarrow 1-2z; \quad \frac{1}{\xi} \leftrightarrow 1-2\zeta; \quad t \leftrightarrow W^2$$

- Duality in the spirit of R. Dolen, D. Horn, C. Schmid'67. GPDs are presented as infinite series of t-channel Regge exchanges M. Polyakov'98:



$$\langle \pi(p')|\,\hat{O}\,|\pi(p)\rangle \sim \text{ Crossing of } \sum_{R_J} \sum_{\substack{\text{polarization}\\ \text{of } R_J}} \frac{1}{t - M_{R_J}^2}$$

$$\times \underbrace{\langle \pi(p')\pi(-p)|R_J\rangle}_{R_J N\bar{N} \text{ effective vertex}} \; \underbrace{\langle R_J|\,\hat{O}\,|0\rangle}_{\text{F.T. of DA of } R_J} .$$

- Expansion in the t-channel PW:

$$\cos\theta_t = \frac{s-u}{\sqrt{1-\frac{4\,m^2}{t}}\;(Q^2+t)} = \frac{1}{\xi\sqrt{1-\frac{4m^2}{t}}} + O\!\left(\frac{1}{Q^2}\right),$$

Combinations of nucleon GPDs suitable for PW expansion in the t-channel PW:

$$H^{(E)} \equiv H + \frac{t}{4M_N^2} E : \quad P_l\left(\frac{1}{\xi}\right) \qquad H^{(M)} \equiv H + E : \; P_l'\left(\frac{1}{\xi}\right)$$

$$\tilde{H}^{(PS)} \equiv \tilde{H} + \frac{t}{4M_N^2} \tilde{E} : \quad P_l\left(\frac{1}{\xi}\right) \qquad \tilde{H} : P_l'\left(\frac{1}{\xi}\right)$$

E.g. unpolarized singlet ($C = +1$) quark GPDs ($H_+^q(x,0,0) = q(x) + \bar{q}(x)$) and unpolarized gluon GPDs ($H^g(x,0,0) = xg(x)$)

$$H_+^{q\,(E,M)}(x,\xi,t) = 2 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{nl}^{q\,(E,M)}(t)\, \theta\left(1 - \frac{x^2}{\xi^2}\right) \left(1 - \frac{x^2}{\xi^2}\right) C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right) \begin{Bmatrix} P_l\left(\frac{1}{\xi}\right) \\ \frac{1}{\xi} P_l'\left(\frac{1}{\xi}\right) \end{Bmatrix}$$

$$H^{g\,(E,M)}(x,\xi,t) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{nl}^{g\,(E,M)}(t)\, \theta\left(1 - \frac{x^2}{\xi^2}\right) \left(1 - \frac{x^2}{\xi^2}\right)^2 C_{n-1}^{\frac{5}{2}}\left(\frac{x}{\xi}\right) \begin{Bmatrix} \xi P_l\left(\frac{1}{\xi}\right) \\ P_l'\left(\frac{1}{\xi}\right) \end{Bmatrix}$$

Dual parametrization: summing up the formal series

How to construct the convolution kernels? $H(x,\xi,t)=\sum_{k=0}^{\infty}\int_0^1 dy K^{(k)}(x,\xi,y)Q_k(y,t)\,.$

- Mellin moments of $Q_k(y,t)$ generate the generalized F.F. B_{nl}:

$$B_{n\;n+1-k}(t)=\int_0^1 dy y^n Q_k(y,t)\,.$$

- M. Polyakov and A. Shuvaev'02 (see also M. Polyakov and KS'08):

$$K^{(k)}(x,\xi,y)=\mathrm{disc}_{z=x}F^{(k)}(z,\xi,y)\,,\qquad \text{where}$$

$$F^{(k)}(z,\xi,y)=\frac{1}{y}\left(1+y\frac{\partial}{\partial y}\right)\int_{-1}^{1}ds\xi^k\frac{z_s^{1-k}}{\sqrt{z_s^2-2z_s+\xi^2}}\,,\quad z_s\equiv 2\frac{z-\xi s}{(1-s^2)y}.$$

Two ways to compute the discontinuity:

1. Expand in powers of $\frac{1}{z_s}$ and employ Rodriguez formula for Gegenbauer polynomials $\Rightarrow$ formally recover conformal PWE for GPD.
2. Consider the discontinuity due to the cut $1-\sqrt{1-\xi^2}<z_s<1+\sqrt{1-\xi^2}$ (and from poles at $z_s=0$ for $k\geq 2$) $\Rightarrow$ analytical expressions for the convolution kernels in terms of elliptic integrals.

Basic properties

- GPD is presented as a convolution over y of convolution kernels with the set of forward-like functions Q_k (G_k for gluon GPDs).
- Scale dependence of Q_k, G_k is given by DGLAP equations.
- GPDs satisfy polynomiality property and the support property.
- The D-term is the natural ingredient of the dual parametrization.
- The limit $\xi \to 0$. The forward-like function $Q_0^{(E)}(x, t=0) \equiv Q_0(x)$ and $G_0^{(E)}(x, t=0) \equiv G_0(x)$ are expressed as

$$\checkmark \quad Q_0(x) = q(x) + \bar{q}(x) - \frac{x}{2}\int_x^1 \frac{dy}{y^2}\,(q(y) + \bar{q}(y))\,;$$

$$\checkmark \quad G_0(x) = 9x^2 \int_x^1 \frac{dy}{y^3} g(y) - 3x \int_x^1 \frac{dy}{y^2} g(y)\,.$$

- "Minimalist" model for GPDs which takes into account only Q_0, G_0. does not describe properly the DVCS data! Thus more conformal PW are to be included!
- A principle allowing to take into account only a finite number of conformal PWs (i.e. Q_k, G_k)?

Convolutions with hard kernels

- Our final goal is the computation of the Compton F.Fs.
- The elementary amplitude is the natural building block.

$$A(\xi) = \int_0^1 dx H(x,\xi) \left[\frac{1}{\xi - x - i0} - \frac{1}{\xi + x - i0} \right] = 4 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{n\,l} P_l \left(\frac{1}{\xi} \right) ;$$

$$\mathrm{Im} A(\xi) = 2 \int_{\frac{1-\sqrt{1-\xi^2}}{\xi}}^{1} \frac{dx}{x} N(x) \frac{1}{\sqrt{\frac{2x}{\xi} - x^2 - 1}} .$$

- GPD quintessence $N(x) = \sum_{\nu=0}^{\infty} x^{2\nu} Q_{2\nu}(x)$

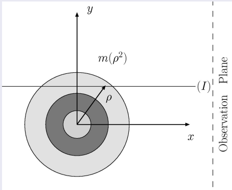
♠ A problem reported for less trivial kernels (e.g. for α_s correction to CFF:)

$$\int_0^1 dx\, C(x,\xi) H(x,\xi) = 4 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \sum_{\substack{l=0 \\ \text{even}}}^{n+1} B_{n\,l} \mathcal{C}(n) P_l \left(\frac{1}{\xi} \right)$$

In order to sum up the PW expansion for CFF one has to build an integral transformation

$$\int_0^1 dy\, y^{k+l-1} \left(\mathcal{K}_{\mathcal{C}}(y, z_1, ..., z_p), Q_k(z_p) \right) = \mathcal{C}(k+l-1) B_{nl}$$

Abel transform tomography



The observer at ∞ looking along a line parallel to the x-axis a distance y above the origin sees the projection:

$$a(y^2) = \int_{-\infty}^{\infty} dx\, m(\rho^2) = \int_{y^2}^{\infty} d\rho^2 \frac{m(\rho^2)}{\sqrt{\rho^2 - y^2}}$$

- M. Polyakov'07: with the help of Joukowski conformal map $\frac{1}{w} = \frac{1}{2}\left(x + \frac{1}{x}\right)$ it is possible to present the relation between $\mathrm{Im}A(\xi)$ and GPD quintessence $N(x)$ in the form of the Abel integral equation.
- The inverse transform for $N(x)$:

$$N(x) = \frac{1}{\pi} \frac{x(1-x^2)}{(1+x)^{\frac{3}{2}}} \int_{\frac{2x}{1+x^2}}^{1} \frac{d\xi}{\xi^{\frac{3}{2}}} \frac{1}{\sqrt{\xi - \frac{2x}{1+x^2}}} \left\{ \frac{1}{2}\mathrm{Im}A(\xi) - \xi \frac{d}{d\xi}\mathrm{Im}A(\xi) \right\}.$$

- The information on GPDs from the amplitude of hard exclusive process can be quantified in terms of a GPD quintessence function and the value of the D-form factor.

- The key role is played by the expansion of GPD $H(x,\xi)$ in powers of ξ around the point $\xi=0$ with fixed x ($x>\xi$).

$$
\begin{aligned}
H(x,\xi) &= H^{(0)}(x)+\xi^2 H^{(2)}(x)+\xi^4 H^{(4)}(x)+\ldots \\
&= Q_0(x)+\frac{\sqrt{x}}{2}\int_x^1 \frac{dy}{y^{3/2}}\,Q_0(y)+\xi^2\Bigg[-\frac{1-x^2}{4x}\frac{\partial}{\partial x}Q_0(x)+ \\
&\frac{1}{32}\int_x^1 dy\,Q_0(y)\left\{\frac{1}{y}\left(3\sqrt{\frac{x}{y}}+3\sqrt{\frac{y}{x}}\right)+\frac{1}{y^3}\left(3\sqrt{\frac{y}{x}}-\left(\frac{y}{x}\right)^{\frac{3}{2}}\right)\right\} \\
&+\frac{1}{4}Q_2(x)+\frac{3}{32}\int_x^1 dy\,Q_2(y)\frac{1}{y}\left(\frac{1}{2}\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}+\frac{5}{2}\left(\frac{y}{x}\right)^{\frac{3}{2}}\right)\Bigg]+O(\xi^4)
\end{aligned}
$$

- Up to the order $\xi^{2\mu}$ this expansion involves only $Q_{2\nu}(x)$ with $\nu\le\mu$
- Assume that the expansion of GPD $H(x,\xi)$ around $\xi=0$ for $x>\xi$ calculated in the framework of a certain parametrization/phenomenological model is known: $H(x,\xi)=\phi_0(x)+\phi_2(x)\xi^2+\phi_4(x)\xi^4+O(\xi^6)$,
  with $\phi_{2\nu}(x)=\frac{1}{(2\nu)!}\frac{\partial^{2\nu}}{\partial\xi^{2\nu}}H(x,\xi)_{\xi=0}$.
- Using this expansion we are about to recast any particular model for $H(x,\xi)$ in the framework of the dual parametrization and determine the corresponding functions $Q_{2\nu}(x)$ order by order.

Reparametrization procedure II

- For $Q_0(x)$ the usual expression is recovered.
- The result for $Q_2(x)$ reads:

$$Q_2(x) = \frac{2(1-x^2)}{x^2}\, q(x) + \frac{(1-x^2)}{x}\, q'(x) + \int_x^1 dy \left(\frac{-15\,x}{4\,y^4} - \frac{3}{2\,y^3} + \frac{5\,x}{4\,y^2} \right) q(y)$$
$$+ 4\,\phi_2(x) - \int_x^1 dy\, \phi_2(y) \left(\frac{15\,x}{4\,y^2} + \frac{3}{2y} + \frac{3}{4\,x} \right).$$

- The derivation of results for Q_4, Q_6, etc is straightforward.

Some lessons

- A problem reported! Assume $q(x) \sim \frac{1}{x^\alpha}$ with $\alpha \approx 1$. Then $Q_2(x) \sim \frac{1}{x^{2+\alpha}}$ and in general $Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}}$. This leads to the possible divergences of

$$B_{2\nu-1\ \ 0} = \int_0^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x)\,.$$

- Note that $B_{2\nu-1\ \ 0}$ are the lowest order Mellin moments of the forward like functions $Q_{2\nu}$ with $\nu > 0$ relevant for the calculation of GPDs. In the DVCS amplitude these $B_{2\nu-1\ \ 0}$ contribute only into the D form factor. This has deep consequences.

Dual parametrization v.s. Radyushkin DD Ansatz I

Reparametrization procedure allows to establish the link between the dual parametrization of GPDs and RDDA, Radyushkin'97.

GPD is obtained as a one dimensional section of a two-variable double distribution f^q:

$$H^q(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, \delta(x-\beta-\alpha\xi) \, f^q(\beta,\alpha) + D\text{-term}$$

RDDA: $f^q(\beta,\alpha) = \mathrm{h}(\beta,\alpha) q(\beta)$.

$$\mathrm{h}^{(b)}(\beta,\alpha) = \frac{\Gamma(2b+2)}{2^{2b+1}\Gamma^2(b+1)} \frac{[(1-|\beta|)^2-\alpha^2]^b}{(1-|\beta|)^{2b+1}}$$

- Several first forward like functions $Q_{0,2,4}$ that reexpress Radyushkin DD Ansatz in the framework of the dual parametrization were computed.
- A way to compare: assume power-like asymptotic behavior of $q(x)$ for small x: $q(x) \sim \frac{1}{x^\alpha}$ with $1 < \alpha < 2$ and compare $\mathrm{Im} A(\xi)$ for $\xi \sim 0$.

Dual parametrization v.s. Radyushkin DD Ansatz II

$\mathrm{Im}A(\xi)$ for $\xi \sim 0$ from $Q_{0,2,4}(x)$:

$$\mathrm{Im}A^{(0)}(\xi) + \mathrm{Im}A^{(2)}(\xi) + \mathrm{Im}A^{(4)}(\xi) + \ldots$$
$$\sim \frac{2^{\alpha+1}}{\xi^{\alpha}} \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+2)} \left\{1 + (\alpha-b)\, c_2(\alpha,b) + (\alpha-b)\,(\alpha-b+1)\, c_4(\alpha,b) + \ldots\right\} .$$

$$\mathrm{Im}A_{DD}(\xi) \sim \frac{2^{2b+1-\alpha}}{\xi^{\alpha}} \frac{\Gamma(\frac{1}{2})\Gamma(b+\frac{3}{2})\Gamma(1+b-\alpha)}{\Gamma(2+2b-\alpha)}$$

- For $\alpha = b$ the coefficients in front of leading singular term of $\mathrm{Im}A_{DD}(\xi)$ and $\mathrm{Im}A^{(0)}(\xi)$ coincide. For small ξ the minimalist dual model is equivalent to RDDA with $b = 1$.
- For $b = \alpha + M$, $M > 0$, integer, it suffices to take account of a finite number of forward-like functions $Q_{2\nu}$ with $\nu \leq M$ obtained using the reparametrization procedure to reproduce the leading small-ξ asymptotic behavior of $\mathrm{Im}A_{DD}(\xi)$.
- The two parametrizations result in distinct behavior of $\mathrm{Im}A(\xi)$ for $\xi \sim 1$. One has to sum up all partial waves in the dual parametrization in order to reproduce $\sim (1-\xi)^b$ behavior of $\mathrm{Im}A(\xi)$ in RDDA.

Skewness effect for $H^{q\,g\,(E,M)}$

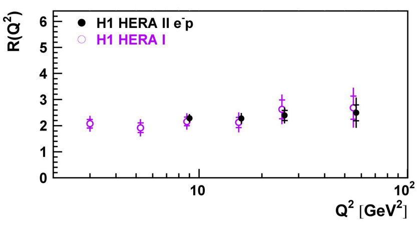
- Assume that $q^{(E,M)}(x) \sim 1/x^{\alpha^q}$; $g^{(E,M)}(x) \sim 1/x^{\alpha^g}$.

Skewness effect in the "minimalist" dual model equals conformal ratio (K. Kumericki, D. Mueller and K. Passek-Kumericki'08, 09)

$$r_{Q_0}^{q\,(E,M)} \equiv \left.\frac{H^{q\,(E,M)}(\xi,\xi)}{H^{q\,(E,M)}(\xi,0)}\right|_{\xi\sim 0} \simeq \frac{2^{\alpha^q}\Gamma(\alpha^q+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\alpha^q)} \approx 3/2 \quad \text{for} \quad \alpha^q \approx 1\,;$$

$$r_{G_0}^{g\,(E,M)} \equiv \left.\frac{H^{g\,(E,M)}(\xi,\xi)}{H^{g\,(E,M)}(\xi,0)}\right|_{\xi\sim 0} \simeq \frac{2^{\alpha^g+1}\Gamma(\alpha^g+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(3+\alpha^g)} \approx 1 \quad \text{for} \quad \alpha^g \approx 1\,.$$

Skewness effect from H1:



$$\mathbf{R} = 2^{\alpha_q} r^q \sim \frac{\sqrt{\sigma_{DVCS}}}{\sigma_{DIS}}$$

The observable ratio $R(\mathrm{Q}^2)$ for fixed $W = 82$ GeV. The Figure is taken from H1'07.

Some lessons

- In order to describe the data the dual parametrization model should include some additional forward like functions $Q_{2\nu}^{(E,M)}$ with $\nu > 0$. These functions should be singular enough in order to make influence on the small ξ asymptotic behavior of $\mathrm{Im}A(\xi)$.
- Same problem in other words. Conformal partial wave expansion written as a Mellin-Barnes integral K. Kumericki, D. Mueller and K. Passek-Kumericki'08 employ

$$m_j(\xi) = \xi^{j+1} \sum_{J=J_{\min}}^{j+1} \frac{h_J}{J-\alpha(t)} P_J\left(\frac{1}{\xi}\right) .$$

In addition to the LO $\mathrm{SO}(3)$ partial wave $(J = j+1)$ the next to leading $\mathrm{SO}(3)$ partial wave should be included. Then it turns out possible to fit the small-x_{Bj} experimental data to a reasonable accuracy.

Seems to be a problem:

- In order to contribute to the leading small-ξ singular behavior of $\mathrm{Im}A(\xi)$:

$$Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}} .$$

- This leads to divergencies of generalized form factors $B_{2\nu-1\;0}$.
- These divergent generalized form factors contribute only into the D-form factor.

Analytical properties

✓ Once subtracted dispersion relation in $\omega = \frac{1}{\xi}$ for the elementary amplitude reads (*e.g.* Teryaev'05):

$$A(\xi) = 4D^q + \frac{1}{\pi}\int_0^1 d\xi' \left(\frac{1}{\xi - \xi' - i\epsilon} - \frac{1}{\xi + \xi' - i\epsilon} \right) \mathrm{Im}A(\xi' - i\epsilon)\,.$$

Common wisdom:

- The subtraction constant in a dispersion relation presents an independent quantity, which cannot be fixed just with help of the information on the discontinuities of the amplitude. In order to determine the value of the subtraction constant one has to attain certain additional information on the amplitude under consideration.

A way to proceed:

- D. Mueller et al.: fix the value of the subtraction constant assume analytical properties in j of combinations of coefficients $h_{2\nu}^{(2\nu+j)}$ at powers of ξ of Mellin moments of GPD.

$$\int_0^1 dx\, x^N H_+(x,\xi) = h_0^{(N)} + h_2^{(N)}\xi^2 + \ldots + h_{N+1}^{(N)}\xi^{N+1} \quad (N = 1,\, 3,\, \ldots)\,.$$

GPD sum rule

- Dispersion relation together with the definition of the LO amplitude O. Teryaev'05, I. Anikin and O. Teryaev'07 :

$$\int_0^1 dx \left(\frac{1}{\xi - x} - \frac{1}{\xi + x} \right) [H_+(x,\xi) - H_+(x,x)] = 4D^q \, .$$

- Expansion in in powers of $\frac{1}{\xi}$ + polynomiality property $\Rightarrow$ a family of sum rules:

$$\sum_{\nu=1}^{\infty} h_{2\nu}^{(2\nu+j)} = \int_0^1 dx \, x^j \, [H_+(x,x) - H_+(x,0)] \, , \quad \text{with} \quad j = 1, 3, \ldots \, .$$

- Subtraction constant can be fixed:

$$2D^q = \sum_{\nu=1}^{\infty} h_{2\nu}^{(2\nu-1)} = \lim_{j \to -1} \left\{ \int_0^1 dx \, x^j \, [H_+(x,x) - H_+(x,0)] \right\} \, ,$$

Analytical regularization

- Compute for large positive j. Then analytically continue to $j = -1$
- This is precisely a so-called analytic (or canonical) regularization ($1 < \alpha < 2$):

$$\int_{(0)}^1 dx \, \frac{f(x)}{x^{1+\alpha}} = \int_0^1 dx \, \frac{1}{x^{1+\alpha}} \left[f(x) - f(0) - x f'(0) \right] - \frac{f(0)}{\alpha} - \frac{f'(0)}{\alpha - 1} \, .$$

- Restrict the class of functions *e.g.* (I. Gelfand and G. Shilov'64):

$$z^{2\nu}Q_{2\nu}(z),\ N(z),\ \mathrm{Im}A(z) \in \left\{ F: \quad F(z) = \sum_{r=1}^{R} \frac{1}{x^{\alpha_r}} f_r(z) \right\} ,$$

with finite R.

- The subtraction constant can be fixed according to:

$$2D^q = \int_{(0)}^{1} dx \frac{1}{x} \left[H_+(x,x) - H_+(x,0) \right] .$$

How this applies for the dual parametrization:

$$D^q = \int_0^1 \frac{dx}{x} Q_0(x) \left(\frac{1}{\sqrt{1+x^2}} - 1 \right) + \int_{(0)}^{1} \frac{dx}{x} \left[N(x) - Q_0(x) \right] \frac{1}{\sqrt{1+x^2}} .$$

- This suggests the use of analytic regularization:

$$B_{2\nu-1\ \ 0} = \int_{(0)}^{1} \frac{dx}{x} x^{2\nu} Q_{2\nu}(x) .$$

- The possibility to fix the D-form factor strongly relies on the postulated analyticity of Mellin moments of GPDs in Mellin space.
- Once this requirement is lifted the D-term may introduce an independent contribution into $\mathrm{Re}A(\xi)$.
- Adding of a supplementary D-term $\theta(1-\frac{x^2}{\xi^2})\,\delta D\left(\frac{x}{\xi}\right)$ with the Gegenbauer expansion:

$$\delta D(z) = (1-z^2) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \delta d_n \, C_n^{\frac{3}{2}}(z)$$

to a GPD is equivalent to an introduction of the *non analytic* contributions to the forward-like functions in the framework of the dual parametrization:

$$x^{2\nu} Q_{2\nu}(x) \longrightarrow x^{2\nu} Q_{2\nu}(x) + 2\delta d_{2\nu-1} \, x\delta(x);$$

- Such situation occurs in certain dynamical models. *E.g.* pion GPD in nonlocal chiral quark model. See K.S.'08
- This results in terms "invisible" for Abel tomography like $\xi\delta(\xi)$ for $\mathrm{Im}A(\xi)$.

Check of analyticity assumptions?

✓ The value of the D form factor is fixed by the small-x_{Bj} behavior of σ_{DVCS}. Consider a toy example

- Alter the leading small-ξ asymptotic behavior of $\mathrm{Im} A(\xi)$ for small-ξ in order to match well with H1 HERA and ZEUS data.
- The contribution of $N(x) - Q_0(x)$ should make no influence for relatively large values of ξ where according to the result of M. Polyakov and M. Vanderhaeghen' 08 the minimalist model provide the satisfactory description of *e.g.* the Jlab/HallA data.
- Advantage of the dual parametrization: one can model directly $\mathrm{Im} A$. $\mathrm{Re} A$ satisfying dispersion relation can be computed from N restored by the Abel tomography.

$$\mathrm{Im} A^{Q_0}(\xi, t) \sim c^q \, \frac{2^{a(t)+1}}{\xi^{a(t)}} \frac{\Gamma(\frac{1}{2})\Gamma(a(t) + \frac{3}{2})}{\Gamma(a(t) + 2)} \equiv C^{Q_0}(t) \frac{1}{\xi^{a(t)}} \,,$$

where $a(t) = \alpha + \alpha' t$. Then try

$$\mathrm{Im} A^{N - Q_0}(\xi, t) = C^{N - Q_0}(t) \frac{1}{\xi^{a(t)}} (1 - \xi)^{\beta} \,.$$

- Calculation of the D form factor. Analyticaly regularized contribution sensitive to small-ξ behavior dominates D.

$$D^{N-Q_0}(t) = \int_{(0)}^{1} \frac{d\xi}{\xi} \mathrm{Im} A^{N-Q_0}(\xi, t) = C^{N-Q_0}(t) \mathrm{B}(1+\beta, -a(t))\,.$$

- Ties together the t and ξ dependence of GPDs.
- For $1 < \alpha < 2$ there are two "tachion" poles at $t = -\frac{\alpha}{\alpha'}$ and $t = -\frac{\alpha-1}{\alpha'}$.
- In order to get rid of "tachion" contributions into the D form factor:

$$C^{N-Q_0}(t) = C^{N-Q_0}(0) \left(t + \frac{\alpha}{\alpha'}\right) \left(t + \frac{\alpha-1}{\alpha'}\right) \frac{\alpha'^2}{\alpha(\alpha-1)}\,.$$

Conclusions

1. The dual parametrization represents a way of handing conformal PW expansion of GPDs. To large extent it is equivalent to Shuvaev transform and Mellin-Barnes type integral based techniques.
2. Simple generalization for both quark and gluon GPDs (unpolarized, polarized and in principle helicity flip) of spin-$\frac{1}{2}$ hadrons.
3. Basic theoretical requirements hold for GPDs in the dual representation.
4. The parametrization possess several useful features useful for model builders: reparametrization procedure, Abel transform tonmography, etc. But still unable to compute α_s corrections for CFF in a closed form.
5. For small x_{Bj} the minimalist dual model is equivalent to RDDA with $b=1$ (and leading $\mathrm{SO}(3)$ PW approximation for D. Müllers et al. approach).
6. The forward-like functions $Q_{2\nu}(x)$ with $\nu \geq 1$ may contribute to the leading singular small-x_{Bj} behavior of the imaginary part of DVCS amplitude. This makes the small-x_{Bj} behavior of $\mathrm{Im}A^{DVCS}$ independent of the asymptotic behavior of PDFs.
7. Assuming analyticity of Mellin moments of GPDs we are able to fix the value of the D-form factor in terms of the GPD quintessence function and the forward-like function $Q_0(x)$. "Duality property" of GPDs is respected.
8. The value of the D form factor is fixed by the small-x_{Bj} behavior of σ_{DVCS}.

On the possible non analytic contributions II

$l = 0$ fixed pole contribution

- The analyticity of Mellin moments in Mellin space can be absent due to the so-called fixed pole singularity at $j = -1$. In the Mellin space this kind of singularity reveals itself as a term proportional to a Kronecker δ_{-1j} which is *non analytic* in j.
- The existence of a fixed pole at $j = -1$ (i.e. angular momentum $l = 0$) in the case of forward Compton scattering amplitude was reasoned with the Regge theory inspired argumentation in and revealed in the experimental measurements (see C. Damashek'70, C. Dominguez'70).
- Nevertheless, according to J. Cornwall et al. '70,71, S. Brodsky et al. '71,73 via a subtracted sum rule the fixed pole contribution can be related to the imaginary part of Compton amplitude. Once generalized for the case of DVCS these considerations would also imply no independent D-term contribution into DVCS amplitude (see S. Brodsky'08).