Spin structure function $g_1$ at small $Q^2$ and the recent COMPASS data

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talk based on results obtained in collaboration with
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Outlook of theoretical results on $g_1$

The spin-dependent part of the hadronic tensor is parameterized by two structure functions:

$$ W_{\mu\nu}^{\text{spin}} = \frac{m}{pq} i\epsilon_{\mu\nu\lambda\rho} q_\lambda S_\rho g_1(x, Q^2) + \left( S_\rho - \frac{S q}{pq} p_\rho \right) g_2(x, Q^2) $$

where $m$, $p$ and $S$ are the hadron mass, momentum and spin; $q$ is the virtual photon momentum ($Q^2 = -q^2 > 0$). Both of the functions depend on $Q^2$ and $x = Q^2 / 2pq$, $0 < x < 1$. 
When $2pq$ is large compared to the mass scale, one can use the factorization:

$$W_{\mu\nu}^{spin} = \Phi_{\text{quark}} + \Phi_{\text{gluon}}$$

Probability to find quark in the hadron

Probability to find gluon in the hadron
DIS off quark and gluon can be studied with perturbative QCD.

Probabilities to find a quark/gluon involve non-perturbative QCD. There is no a regular analytic way to calculate them. Instead, they are substituted by initial quark and gluon densities defined from experimental data at $x \sim 1$ and $Q^2 \sim 1$ GeV$^2$

So, the conventional form of the hadronic tensor is:

$$W_{\mu\nu} = W_{\mu\nu}^{\text{quark}} \otimes \delta q + W_{\mu\nu}^{\text{gluon}} \otimes \delta g$$
Kinematical regions to cover:

- **DGLAP proper region**
- **COMPASS region**
- **HERA region**

- **Region A**:
  - $x \leq 1$
  - $Q^2 \gg \mu^2$

- **Region B**:
  - $x \ll 1$
  - $Q^2 \gg \mu^2$

- **Region C**:
  - $x \ll 1$
  - $Q^2 \leq \mu^2$

- **Region D**:
  - $x \leq 1$
  - $Q^2 \leq \mu^2$

$\mu^2 \approx \text{few GeV}^2$
The Standard instrument for theoretical investigation of DIS is DGLAP

In particular, Altarelli-Parisi, Gribov-Lipatov, Dokshitzer

\[ g_1^{NS}(x, Q^2) = \left( e_q^2 / 2 \right) C_{NS}(x, y) \otimes \Delta q(y, Q^2) \]

Expression for the singlet \( g_1 \) is similar, though more involved. It includes two coefficient functions, four anomalous dimensions and, in addition to the quark distribution, the gluon distribution.
DGLAP evolution equations for the singlets

\[
\frac{d\Delta q}{d \ln Q^2} = \frac{\alpha_s}{2\pi} P_{qq} \otimes \Delta q + \frac{\alpha_s}{2\pi} P_{qg} \otimes \Delta g
\]

\[
\frac{d\Delta g}{d \ln Q^2} = \frac{\alpha_s}{2\pi} P_{gq} \otimes \Delta q + \frac{\alpha_s}{2\pi} P_{gg} \otimes \Delta g
\]

splitting functions

Mellin transform of the splitting functions = anomalous dimensions
Applying the Mellin transform, obtain a simpler expression for $g_1^{NS}$:

$$g_1^{NS}(x, Q^2) = \frac{e_q^2}{2}$$

$$\int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left( \frac{1}{x} \right)^\omega C(\omega) \delta q(\omega) \exp \left[ \int \frac{d^2 k_\perp}{\mu^2} \gamma(\omega, \alpha_s(k^2_\perp)) \right]$$

Coefficient function

Initial quark density

Anomalous dimension

Non-Pert QCD

Pert QCD
$g_1$ at $x << 1$ and $Q^2 >> m^2$

$x$-evolution of $Dq$ with coefficient function

$Q^2$-evolution of $d_q$ with anomalous dimension

$Dq$ at $x \sim 1$ and $Q^2 >> m^2$

$Dq$ at $x \sim 1$ and $Q^2 \sim m^2$ defined from fitting exp data

Starting point of $Q^2$-evolution

evolved quark density
In DGLAP, coefficient functions and anomalous dimensions are known with LO and NLO accuracy, often at integer $w = n$

$$C(\omega) = 1 + (\alpha_s(Q^2)/2\pi) C^{(1)}(\omega) + \ldots$$

$$\gamma(\omega) = (\alpha_s(Q^2)/4\pi) \gamma^{(0)}(\omega) + (\alpha_s(Q^2)/2\pi)^2 \gamma^{(1)}(\omega) + \ldots$$

One can say that DGLAP includes both Science and Art:
g_1 at small x and large Q^2

starting point

Region B

x-evolution, total resummation of

(\alpha_s \ln^2(1/x)^k, (\alpha_s \ln(1/x))^k

DGLAP does not account for the total resummation of logs of x and from theoretical grounds cannot be used in Region B
In practice SA solves this problem through introducing singular fits for initial parton densities, they cause a fast growth at small $x$ and thereby mimic the resummation. **Week point:** no theoretical grounds

Altarelli-Ball-Forte-Ridolfi, Blumlein-Botcher, Leader-Sidorov-Stamenov, Hirai et al

In the literature, there are different fits for initial parton densities. For example,

\[
\delta q = Nx^{-\alpha} \left[ (1 - x)^\beta (1 + \gamma x^\delta) \right] \\
\delta q = N \left[ \ln^\alpha \left( \frac{1}{x} \right) + \gamma x \ln^\beta \left( \frac{1}{x} \right) \right]
\]

Altarelli-Ball-Forte-Ridolfi,

Parameters $N, \alpha, \beta, \gamma, \delta$ should be fixed from experiment

**Alternative, Straightforward Way:** Total resummation of leading logs of $x$
As value of the cut-off is not fixed, one can evolve the structure functions with respect to $m$ the name of the method:

**Infra-Red Evolution Equations (IREE)**

Highlights of the history of the method

- Analyses of two-particle cuts in Regge kinematics
  
  Gribov

- Factorization of photons with small transverse momenta
  
  Gribov

- Infrared cut-off in the transverse momentum space
  
  Lipatov

- Quark-quark scattering amplitudes
  
  Kirschner-Lipatov

- Generalization of Gribov bremsstrahlung theorem to QCD, inelastic quark form factors
  
  Ermolaev-Fadin -Lipatov

- QCD inelastic processes in Regge kinematics
  
  Ermolaev-Lipatov

- Applications to Polarized Deep-Inelastic scattering
  
  Bartels-Ermolaev -Manaenkov-Ryskin- Greco-Troyan
Expression for the singlet $g_1$ at large $Q^2$: $Q^2 > \mu^2; \mu \approx 5$ GeV

\[ g_1^S = \frac{< e_q^2 >}{2} \int \frac{d\omega}{2\pi i} \left( \frac{1}{x} \right)^\omega \left[ F_q \delta q + F_g \delta g \right] \]

\[ F_q = C_q^{(+)} \left( \frac{Q^2}{\mu^2} \right)^{\Omega^{(+)}} + C_q^{(-)} \left( \frac{Q^2}{\mu^2} \right)^{\Omega^{(-)}} \]

\[ F_g = C_g^{(+)} \left( \frac{Q^2}{\mu^2} \right)^{\Omega^{(+)}} + C_g^{(-)} \left( \frac{Q^2}{\mu^2} \right)^{\Omega^{(+)}} \]

$C_{q,g}^{(+)}(\omega), C_{q,g}^{(-)}(\omega), \Omega^{(+)}(\omega), \Omega^{(+)}(\omega)$ contain leading logs to all orders

coefficient functions anomalous dimensions
Description of $g_1$ in Region C: small $Q^2$ and small $x$:

Generalization of our previous results through the shift

\[ Q^2 \rightarrow Q^2 + \mu^2 \quad \xrightarrow{\text{Infrared cut-off}} \quad x \rightarrow \bar{x} = (Q^2 + \mu^2)/2pq = x + z \]

Similar shifts have been used for DIS structure functions by many authors, however from phenomenological considerations. We do it from analysis of the involved Feynman graphs.
Expression for the singlet $g_1$ at small $Q^2$: $Q^2 < \mu^2; \mu \approx 5 \text{ GeV}$

$$g_1^s = \frac{\langle e_q^2 \rangle}{2} \int \frac{d\omega}{2\pi i} \left( \frac{1}{x + z} \right) \omega \left[ F_q \delta q + F_g \delta g \right]$$

$$F_q = C_q^{(+)} \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega^{(+)}} + C_q^{(-)} \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega^{(-)}}$$

$$F_g = C_g^{(+)} \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega^{(+)}} + C_g^{(-)} \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega^{(+)}}$$

$$x = Q^2 / 2pq, \quad z = \mu^2 / 2pq$$
**COMPASS:** $10^{-1} \text{ GeV}^2 < Q^2 < 3 \text{ GeV}^2$ → DGLAP cannot be used:

$$\ln \left[ \frac{\ln \left( \frac{Q^2}{\Lambda^2} \right)}{\ln \left( \frac{\mu^2}{\Lambda^2} \right)} \right] > 1 \Rightarrow Q^2 \gg \mu^2$$

Our approach is not sensitive to values of $Q^2$, so we can use it

**Prediction 1:** very weak dependence $g_1$ on $x$ at the COMPASS range of $Q^2$ even at very small $x$

when

$Q^2 << \mu^2$

$$x << z \Rightarrow g_1(x+z) \approx g_1(z) + x d g_1(z) / dz + ...$$

$$z = \mu^2 / (2 pq)$$

$$g_1(z) = \left( \frac{< e_q^2 >}{2} \right) \int_{-i\infty}^{i\infty} d\omega \left( \frac{1}{z} \right)^\omega \left[ C_q(\omega) \delta q + C_g(\omega) \delta g \right]$$
so $Q^2$- dependences is flat, even for $x<<1$.

Location of the line cannot be predicted from theoretical grounds because depends on the interplay between the unknown initial quark and gluon densities: at small $Q^2$

$$g_1(z) = \left( \frac{\langle e_q^2 \rangle}{2} \right) \int_{-\infty}^{\infty} \frac{d \omega}{2\pi i} \frac{1}{z} \omega \left[ C_q(\omega) \delta q + C_g(\omega) \delta g \right]$$

Status of this prediction:
Confirmed by COMPASS

are calculated
COMPASS data: $g_1 = 0$ at small $Q^2$ but values of $2pq$ are not specified. It leaves two possibilities:

Case A: $g_1 = 0$ at any value of $2pq$ ($=w$) in the COMPASS $2pq$- range: $30 \text{ GeV}^2 < w < 270 \text{ GeV}^2$

$$\int_{-\infty}^{\infty} d\omega \left( \frac{\mu^2}{w} \right)^{\omega} \left[ C_q(\omega) \delta_q + C_g(\omega) \delta_g \right] \approx 0$$

for all $w$ from the interval $w_{\text{min}} \leq w \leq w_{\text{max}}$
Case B: $g_1=0$ only after averaging over $w$

**Problem:** What expressions for the Initial parton densities to use? Usually they are fixed from phenomenological considerations.

For example, in DGLAP

$$\delta q = 0.4 z^{0.5} (1-z)^3 (1+3z), \quad \delta g = 1.7 z^{-0.5} (1-z)^4 (1+3z)$$

(Altarelli-Ball-Forte-Ridolfi)

Singular at $z \rightarrow 0$
DGLAP needs singular factors to mimic the total resummation of logs. When the resummation is accounted for, they should be dropped, so the fits can be chosen as

\[ \delta q = N_q z^{a_q} (1 - z)^{b_q}, \quad \delta g = N_g z^{a_g} (1 - z)^{b_g}, \]

with all parameters >0, so there are no singularities in the fits. They are supposed to mimic the hadron structure.

**PROBLEM:** in the COMPASS range of \( w, \ z \) is not small: \( 0.5 > z > 0.17 \), so non-logarithmic contributions to the coefficient functions are essential and must be accounted for. We do it in the one-loop approximation, like it is in NLO DGLAP.

**SUGGESTION:** Let us choose the DGLAP-like fits for the initial parton densities, then play with parameters in them.
\[ \delta q = 0.5 \ (1-x)^3 \ x \]
\[ \delta g = 1. \ (1-x)^4 \]

FIG. 1: \( G(z) \) for \( \delta q = 0.5z(1-z)^3 \), \( \delta g = N_g(1-z)^4 \) and for \( N_g = 1.5 \). Thin curve – pure AP. Thick curve – BE+. Dashed curve displays 1-loop DL.

\[ \delta q = 0.5 \ (1-x)^3 \ x \]
\[ \delta g = 2. \ (1-x)^4 \]

FIG. 2: \( G(z) \) for \( \delta q = 0.5z(1-z)^3 \), \( \delta g = N_g(1-z)^4 \) and for \( N_g = 2.5 \). Thin curve – pure AP. Thick curve – BE+. Dashed curve displays 1-loop DL.

\[ \delta q = 0.5 \ (1-x)^3 \ x \]
\[ \delta g = 3. \ (1-x)^4 \]

FIG. 3: \( G(z) \) for \( \delta q = 0.5z(1-z)^3 \), \( \delta g = N_g(1-z)^4 \) and for \( N_g = 3.5 \). Thin curve – pure AP. Thick curve – BE+. Dashed curve displays 1-loop DL.
\[ \Delta q = 0.5 (1 - x)^3 x \]
\[ \Delta g = -1. (1 - x)^4 \]

FIG. 1: \( G(z) \) for \( \delta q = 0.5 z (1 - z)^3 \), \( \delta g = N_z (1 - z)^4 \) and for \( N_z = -1, -0.5 \). Thin curve – pure AP. Thick curve – BE+ .

\[ \Delta q = 0.5 (1 - x)^3 x \]
\[ \Delta g = -0.5 (1 - x)^4 \]

FIG. 2: \( G(z) \) for \( \delta q = 0.5 z (1 - z)^3 \), \( \delta g = N_z (1 - z)^4 \) and for \( N_z = 0, 0.5 \). Thin curve – pure AP. Thick curve – BE+ .
Positive $N_g$, with $N_q/N_g = 0.15$ approx, (Fig 3) are in more agreement with the COMPASS data than negative $N_g$ when DGLAP-like fits for the initial parton densities are used.

Alternatively, let us use the fits with the proton structure neglected. Such fits are proportional to the delta-function:

$$\delta q = N_q \delta(1-z), \quad \delta g = N_g \delta(1-z)$$

Using them brings opposite results:
\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = -0.5 \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{-0.5 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = -0.3 \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{-0.3 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = -0.2 \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{-0.2 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = 0. \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{0 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = 0.5 \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{0.5 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \delta q = 1. \text{DiracDelta}[-1+x] \]
\[ \delta g = 1. \text{DiracDelta}[-1+x] \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

\[ \frac{\delta q}{1. \text{DiracDelta}[-1+x]} \]
\[ \frac{\delta g}{1 \text{DiracDelta}[-1+x]} \]
\[ 10^{-1} \ 10^{-2} \ 10^{-3} \ 10^{-4} \ 10^{-5} \]

FIG. 1: $G(z)$ for $\{\delta q, \delta g\} = \{N_q, N_g\} \delta (1-z)$ and for $N_q = -0.5, -0.3$. Thin curve – pure AP. Thick curve – BE+. Dashed curve displays $G(z) = (1-z) + \left(\frac{\alpha_s^2}{\pi^2} \ln(1/z)\right)[C_F - n_f N_q]$.

FIG. 2: $G(z)$ for $\{\delta q, \delta g\} = \{N_q, N_g\} \delta (1-z)$ and for $N_q = -0.2, 0$. Thin curve – pure AP. Thick curve – BE+. Dashed curve displays $G(z) = (1-z) + \left(\frac{\alpha_s^2}{\pi^2} \ln(1/z)\right)[C_F - n_f N_q]$.

FIG. 3: $G(z)$ for $\{\delta q, \delta g\} = \{N_q, N_g\} \delta (1-z)$ and for $N_q = 0.5, 1.0$. Thin curve – pure AP. Thick curve – BE+. Dashed curve displays $G(z) = (1-z) + \left(\frac{\alpha_s^2}{\pi^2} \ln(1/z)\right)[C_F - n_f N_q]$. 
The plots show that negative $N_g$ are more in agreement with the COMPASS data than positive $N_g$. 
SUMMARY AND OUTLOOK

1. Studying the $x$-dependence of $g_1$ at small $Q^2$ is not of much interest because $g_1$ does not depend on $x$ in this region. Instead, it would be interesting to study the $w$-dependence.

2. In absence of info on the $w$-dependence, the COMPASS result $g_1=0$ can be interpreted in two ways:
   
   (a) $g_1=0$ for any $w$ in the COMPASS $w$-range – incompatible with the both fits we have used
   
   (b) $g_1$, averaged over $w$ is zero- compatible with the both fits, however using different fits leads to opposite conclusions

3. Recovering the $w$-dependence would allow to specify the fits and answer the questions about the sign of the gluon density and to fix the ratio $N_g/N_q$

4. Using another target would allow to measure the non-singlet $g_1$ and define $N_q$