The $\gamma^*\gamma \to \pi^0$ form factor in QCD: a theory appraisal

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$$\int d^4y\, e^{iq_1 y}\langle \pi^0(p)|\,T\{j_\mu^{\rm em}(y) j_\nu^{\rm em}(0)\}|0\rangle = ie^2 \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, F_{\gamma^*\gamma^*\to\pi^0}(q_1^2, q_2^2)$$

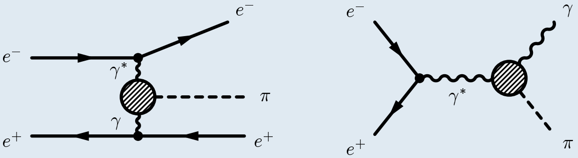


Figure: The pion-photon transition form factor from e^+e^- collisions.

here:

$$q_1^2 = -Q^2 < 0\,, \qquad q_2^2 \to 0$$

ABC of QCD factorization: leading regions

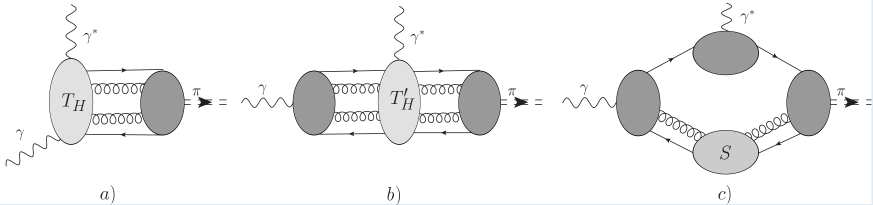


Figure: Schematic structure of the QCD factorization for the $F_{\gamma^*\gamma\to\pi^0}(Q^2)$ formfactor.

$A:$	hard subgraph that includes the both photon vertices	$\frac{1}{Q^2} + \frac{1}{Q^4} + \ldots$
$B:$	real photon is emitted at large distances	$\frac{1}{Q^4} + \ldots$
$C:$	Feynman Mechanism: soft quark spectator	$\frac{1}{Q^4} + \ldots$

- Contributions of regions A, B, C are additive
- All other possibilities lead to exponentially small corrections $\exp[-Q^2]$

Region A: Leading Twist Contribution $\sim 1/Q^2$

- related to OPE for $\mathrm{T}\{j_\mu^{\rm em}(y) j_\nu^{\rm em}(0)\}$ to leading twist accuracy
  — best studied object in QCD
- can be written in factorized form:

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}f_\pi}{3}\int dx\, T_H(x,Q^2,\mu,\alpha_s(\mu))\phi_\pi(x,\mu)$$

$$T_H^{\rm NLO} = \frac{1}{xQ^2}\left\{1 + C_F\frac{\alpha_s(\mu)}{2\pi}\left[\frac{1}{2}\ln^2 x - \frac{x\ln x}{2(1-x)} - \frac{9}{2} + \left(\frac{3}{2}+\ln x\right)\ln\frac{Q^2}{\mu^2}\right]\right\}$$

- NNLO coefficient function known in conformal scheme

Pion Distribution Amplitude (DA)

$$\langle 0|\bar q(0)[0,y]\, \not{y}\gamma_5\, q(y)|\pi^+(p)\rangle \stackrel{y^2=0}{=} i f_\pi\, p\cdot y\int_0^1 dx\, e^{-ixp\cdot y}\,\phi_\pi(x,\mu)$$

- **The ERBL equation**

$$\mu^2 \frac{d}{\mu^2}\phi_\pi(x,\mu) = \frac{\alpha_s(\mu)}{2\pi}\int_0^1 dy\; V(x,y)\phi_\pi(y,\mu)$$

$$V(x,y) = C_F\left[\frac{1-x}{1-y}\left(1+\frac{1}{x-y}\right)\theta(x-y) + \frac{x}{y}\left(1+\frac{1}{y-x}\right)\theta(y-x)\right]_+$$

strongly suggests an expansion in orthogonal polynomials

$$\phi_\pi(x,\mu) = 6x(1-x)\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_n^{(0)}/2\beta_0} a_n(\mu_0)\, C_n^{3/2}(2x-1)\,, \qquad a_0(\mu) = 1$$

— the Gegenbauer expansion

- scale dependence known to NLO in $\overline{MS}$ and NNLO in conformal scheme

- **Relevant nonperturbative information is reduced to the set of numbers,** $a_n(\mu_0)$

— matrix elements of local operators

? How rapidly does the Gegenbauer expansion converge? — later

$a_2(\mu_0)$

Method	$\mu = 1$ GeV	$\mu = 2$ GeV	Reference
LO QCDSR, CZ model	0.56	0.38	[?, ?]
QCDSR	$0.26^{+0.21}_{-0.09}$	$0.17^{+0.14}_{-0.06}$	[?]
QCDSR	0.28 ± 0.08	0.19 ± 0.05	[?]
QCDSR, NLC	0.19 ± 0.06	0.13 ± 0.04	[?, ?, ?]
$F_{\pi\gamma\gamma^*}$, LCSR	0.19 ± 0.05	0.12 ± 0.03 $(\mu = 2.4)$	[?]
$F_{\pi\gamma\gamma^*}$, LCSR	0.32	0.20 $(\mu = 2.4)$	[?]
$F_{\pi\gamma\gamma^*}$, LCSR, R	0.44	0.30	[?]
$F_{\pi\gamma\gamma^*}$, LCSR, R	0.27	0.18	[?]
F_π^{em},LCSR	$0.24 \pm 0.14 \pm 0.08$	$0.16 \pm 0.09 \pm 0.05$	[?, ?]
F_π^{em},LCSR, R	0.20 ± 0.03	0.13 ± 0.02	[?]
$F_{B\to\pi\ell\nu}$, LCSR	0.19 ± 0.19	0.13 ± 0.13	[?]
$F_{B\to\pi\ell\nu}$, LCSR	0.16	0.10	[?]
LQCD, $N_f = 2$, CW	0.329 ± 0.186	0.201 ± 0.114	QCDSF/UKQCD [?]
LQCD, $N_f = 2{+}1$, DWF	0.382 ± 0.143	0.233 ± 0.088	RBS/UKQCD [?]

Table: The Gegenbauer moment $a_2(\mu^2)$. The CZ model involves $a_2 = 2/3$ at the low scale $\mu = 500$ MeV; for the discussion of the extrapolation to higher scales, see Ref. [?]. The abbreviations stand for: QCDSR: QCD sum rules; NLC: non-local condensates; LCSR: light-cone sum rules; R: renormalon model for twist-4 corrections; LQCD: lattice calculation; CW: non-perturbatively $\mathcal{O}(a)$ improved Clover–Wilson fermions; DWF: domain wall fermions.

- **my average for a_2:**

$$a_2(1\ \text{GeV}) = 0.30 \pm 0.15 \qquad a_2(2\ \text{GeV}) = 0.20 \pm 0.07$$

expect 10-15% error from new generation of lattice calculations; precision limited by discretisation errors in operators with derivatives

- **weak constraints on a_4:**
  - QCD SRs with nonlocal condensates (model): $a_4 \sim -0.1$
  - LCSRs for B-meson decays: $a_4 \sim +0.1$
  - Lattice calculation not feasible

- **no information on higher moments**

expect $1 = a_0 > a_2 > a_4 > a_6 > \ldots$

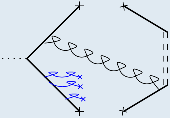
$\Rightarrow$ **models using Gegenbauer expansion truncated at some order**

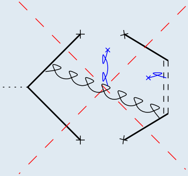
Bakulev, Mikhailov, Stefanis, '04
Mikhailov, Radyushkin, '89–'92

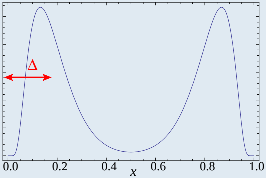
- **QCD sum rules with nonlocal condensates:**

$$\phi_\pi(x) \quad \sim \quad \mathcal{C}_1 \cdot 6x\bar{x}\Big\{1 + \frac{\alpha_s}{3\pi}\Big[5 - \frac{\pi^2}{3} + \ln^2\frac{\bar{x}}{x}\Big]\Big\} + \mathcal{C}_2 \cdot \langle\bar{q}q\rangle^2\Big\{[11\delta(x) + 2\delta'(x)] + (x \leftrightarrow \bar{x})\Big\}$$

Partial resummation of higher-order terms in the OPE







- **Average virtuality of vacuum quarks** $\lambda_q^2 = \langle\bar{q}D^2q\rangle/\langle\bar{q}q\rangle \sim 0.4$ GeV^2

$$\boxed{\Delta = \lambda_q^2/(2M^2) \simeq 0.2}$$

- Strong overlap with $C_4^{3/2}(2x-1)$ $\hookrightarrow a_4 = -0.1, \quad a_{6,8,\ldots} \simeq 0$

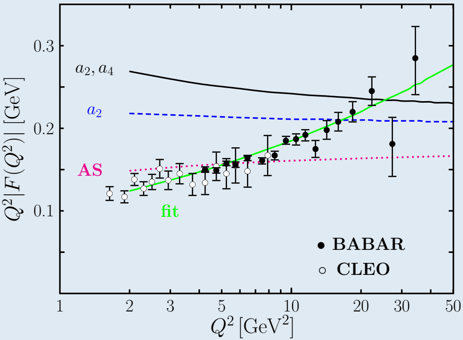
- **Problems:**
  - Approximation theoretically inconsistent. Example: Chernyak hep-ph/0605327
  - Model not tested in other applications and is known to fail in a few cases:
    — parton (quark) distributions VB, Gornicki, Mankiewicz, PRD51 (1995) 6036
    — $B \to \rho \ell \bar{\nu}_\ell$ form factors Ali, VB, Simma, ZPC63 (1994) 437
- **crucial point:**
  - The nonperturbative scale Δ does not appear:

    The series in δ-functions and their derivatives gets smeared over the whole interval $0 < x < 1$

    Example: Photon wave function Ball, VB, Kivel, NPB649 (2003) 263

The BABAR puzzle

- **Fixed-order NLO QCD calculation with $\mu = Q$ does not work:**



Input parameters at 1 GeV:

magenta: $a_0 = 1$,
blue: $a_0 = 1$, $a_2 = 0.39$,
black: $a_0 = 1$, $a_2 = 0.39$, $a_4 = 0.24$

- **Predicted scaling behavior not achieved up to $Q^2 \sim 30$ GeV^2 — not expected**

? **Pre-asymptotic effects**

- What are the options?

1. Resummation of higher order perturbative corrections (k_t-factorization)
2. $1/Q^4$ corrections from region A (higher-twist)
3. $1/Q^4$ corrections from region B (photon emission from large distances)
4. $1/Q^4$ corrections from region C (soft overlap of wave functions)

- What methods are available?

1. Direct calculations/estimates/models
2. Dispersion relations and duality (LCSRs)

Region A: (Sudakov) resummation

Botts, Sterman; Li, Sterman

- Basic idea: retain k_t dependence in the hard kernel

$$\frac{1}{xQ^2} \quad \to \quad \frac{1}{xQ^2 + k_t^2}$$

- Large corrections exponentiate in impact parameter space

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) \quad = \quad \frac{\sqrt{2}f_\pi}{3}\int dx \int \frac{d^2b}{2\pi}\widetilde{T}_H(x,\,Q^2,\,b,\,\mu,\,\alpha_s(\mu))\, e^{-\mathcal{S}}\,\phi_\pi(x,\,b_0/b)$$

- Sudakov form factor

$$\mathcal{S} \quad = \quad s(b,xQ) + s(b,(1-x)Q) + 2\int_{b_0/b}^{\mu}\frac{d\mu'}{\mu'}\gamma_q(\alpha_s(\mu'))$$

$$s(b,\,Q) \quad = \quad \frac{1}{4}\int\limits_{b_0^2/b^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\left[\ln\frac{Q^2}{k_\perp^2}\Gamma_{\text{cusp}}(\alpha_s(k_\perp^2)) + \tilde{\Gamma}(\alpha_s(k_\perp^2))\right]$$

- **Implementation involves subtleties which influence numerical outcome**
- **Usually used in combination with some model for soft effects — later**

Region A: Twist-4 Contribution $\sim 1/Q^4$

- related to OPE for $\mathrm{T}\{j_\mu^{\rm em}(y) j_\nu^{\rm em}(0)\}$ to twist-four accuracy
  — well understood

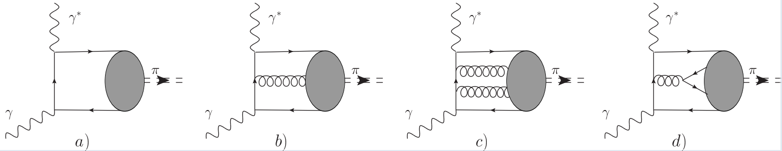


Figure: Twist-4 corrections to the pion transition form factor

- involve twist-4 quark-gluon pion distribution amplitudes

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) \;=\; \frac{\sqrt{2}f_\pi}{Q^2}\left(\frac{1}{3}\int\frac{dx}{x}\phi_\pi(x) - \frac{80}{27}\frac{\delta_\pi^2}{Q^2}\right) \qquad \delta_\pi^2 \simeq 0.2\ \text{GeV}^2$$

- A significant contribution at $Q^2 \sim 1-5$ GeV^2 but unlikely to solve the puzzle

Region B: Photon emission from large distances

- Mainly overlap of twist-three photon and pion wave functions
  — not well understood

- The LO pQCD calculation gives

$$F^{(B)}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}f_\pi}{3}\frac{16\pi\alpha_s\chi\langle\bar{q}q\rangle^2}{9f_\pi^2 Q^4}\int_0^1 dx\,\frac{\phi^p_{3;\pi}(x)}{x}\int_0^1 dy\,\frac{\phi_\gamma(y)}{\bar{y}^2}$$

  which yields a correction

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}f_\pi}{Q^2}\left(\frac{1}{3}\int\frac{dx}{x}\phi_\pi(x) + \frac{0.2\ \text{GeV}^2}{Q^2}\cdot\ln^2\frac{Q^2}{\mu^2_{IR}}\right)$$

- Infrared Divergence signals overlap with soft region C

- (may be) a significant contribution at $Q^2 \sim 1-5$ GeV^2 but unlikely to solve the BABAR puzzle

Region C: Feynman (soft) contribution

- **Overlap of soft wave functions**
  — truly nonperturbative

Musatov–Radyushkin Model

- Use Drell-Yan representation as convolution of light-cone WFs (Brodsky-Lepage)

$$(\varepsilon_\perp \times q_\perp) F^{\bar{q}q}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{f_\pi}{4\pi^3\sqrt{3}} \int_0^1 dx \int d^2k_\perp \frac{(\varepsilon_\perp \times (xq_\perp + k_\perp))}{(xq_\perp + k_\perp)^2 - i\epsilon} \Psi_{\bar{q}q}(x, k_\perp)$$

with a model wave function

$$\Psi_{\bar{q}q}(x, k_\perp) = \frac{4\pi^2}{\sigma\sqrt{6}} \frac{\phi_\pi(x)}{x\bar{x}} \exp\left(-\frac{k_\perp^2}{2\sigma x\bar{x}}\right)$$

to get

$$F^{\rm MR}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}f_\pi}{3} \int_0^1 \frac{dx\,\phi_\pi(x)}{xQ^2} \left[1 - \exp\left(-\frac{xQ^2}{2\bar{x}\sigma}\right)\right]$$

- σ is the width parameter
- using $\sigma = 0.53$ GeV^2 and flat pion DA $\phi_\pi(x) = 1$ can fit the BABAR data !

caveat: $\int dx \int d^2k_t |\Psi_{\bar{q}q}(x, k_\perp)|^2 = \infty$, ?!

- The MR model for the soft contribution:

$$F^{\rm MR}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}f_\pi}{3}\int_0^1 \frac{dx\,\phi_\pi(x)}{xQ^2}\left[1-\exp\left(-\frac{xQ^2}{2\bar{x}\sigma}\right)\right]$$

- The correction is exponentially suppressed for any finite value of x
  — absent in OPE
- Upon integration one obtains

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) \overset{Q^2\to\infty}{=} \frac{\sqrt{2}f_\pi}{Q^2}\left[1-\frac{4\sigma}{Q^2}\right], \qquad \phi_\pi = 6x(1-x)$$

or

$$F_{\gamma^*\gamma\to\pi^0}(Q^2) \overset{Q^2\to\infty}{=} \frac{\sqrt{2}f_\pi}{Q^2}\left[1+\ln\frac{Q^2}{2\sigma}\right], \qquad \phi_\pi = 1$$

- The effect is roughly equivalent to the cutoff of the end-point region $\int_{2\sigma/Q^2}^1 dx\ldots$

Can one estimate soft corrections model-independently?

$\Rightarrow$ Dispersion relations

Region C: Dispersion relations and duality (LCSRs)

Khodjamirian

- The QCD result satisfies an unsubtracted dispersion relation

$$F^{\rm QCD}_{\gamma^*\gamma^*\to\pi^0}(Q^2,q^2) = \frac{1}{\pi}\int_0^\infty ds\,\frac{{\rm Im}F^{\rm QCD}_{\gamma^*\gamma^*\to\pi^0}(Q^2,-s)}{s+q^2}\,.$$

- An effect of soft terms is to correct the spectral density to look more like

$$F_{\gamma^*\gamma^*\to\pi^0}(Q^2,q^2) = \frac{\sqrt{2}f_\rho F_{\gamma^*\rho\to\pi^0}(Q^2)}{m_\rho^2+q^2} + \frac{1}{\pi}\int_{s_0}^\infty ds\,\frac{{\rm Im}F_{\gamma^*\gamma^*\to\pi^0}(Q^2,-s)}{s+q^2}\,.$$

- Duality: assume that above a certain threshold

$${\rm Im}F_{\gamma^*\gamma^*\to\pi^0}(Q^2,-s) \quad = \quad {\rm Im}F^{QCD}_{\gamma^*\gamma^*\to\pi^0}(Q^2,-s) \qquad \text{for} \quad s > s_0$$

- Asymptotic freedom: QCD expression must be correct at $q^2 \to -\infty$, therefore

$$\sqrt{2}f_\rho F_{\gamma^*\rho\to\pi^0}(Q^2) = \frac{1}{\pi}\int_0^{s_0} ds\,{\rm Im}F^{\rm QCD}_{\gamma^*\gamma^*\to\pi^0}(Q^2,-s)\,.$$

- Duality sum rules: use this result to correct the QCD calculation

Leading order example

- QCD calculation

$$F^{\rm QCD}_{\gamma^*\gamma^*\to\pi^0}(Q^2,q^2) = \frac{\sqrt{2}f_\pi}{3}\int_0^1 \frac{dx\,\phi_\pi(x)}{xQ^2+\bar{x}q^2}\,.$$

- LCSR

$$F^{\rm LCSR}_{\gamma^*\gamma\to\pi^0}(Q^2) \quad = \quad \frac{\sqrt{2}f_\pi}{3}\left\{\int_{x_0}^1 \frac{dx\,\phi_\pi(x)}{xQ^2} + \int_0^{x_0}\frac{dx\,\phi_\pi(x)}{\bar{x}m_\rho^2}\right\}, \qquad x_0 = \frac{s_0}{s_0+Q^2}$$

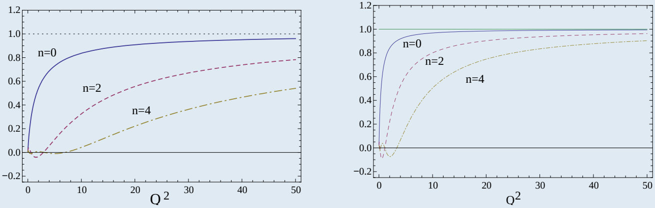
- The difference is a soft correction

Region C: LCSR vs. MR model

- Separate contributions of different Gegenbauer polynomials

$$Q^2 F_{\gamma^*\gamma\to\pi^0}(Q^2) = \sqrt{2} f_\pi \left\{ f_0(Q^2) + a_2 f_2(Q^2) + a_4 f_4(Q^2) + \ldots \right\}$$

…and compare the coefficients $f_n(Q^2)$



MR Model **LCSR**

- A qualitative agreement

Convincing evidence for strong suppression of end-point regions alias contributions of higher Gegenbauer polynomials in pion DA

How good is the Gegenbauer expansion?

- **It has been argued:**

— **BABAR data indicate an "unusual" pion DA $\phi_\pi(x) = 1$ that does not vanish at the end points (Radyushkin)**

— **Gegenbauer expansion does not converge and cannot be applied (Polyakov)**

- **Ist this true?**

$$\phi_\pi^{\rm flat}(x) = 1 = 6x(1-x)\sum_{k=0,2,\ldots} a_k^{\rm flat}\, C_k^{3/2}(2x-1)\,, \qquad a_k^{\rm flat} = \frac{2(2k+3)}{3(k+1)(k+2)}$$

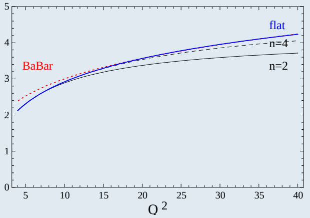
consider approximations to the flat DA as Gegenbauer expansion truncated at order n.

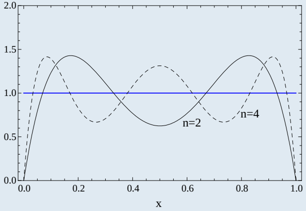
$$\phi_\pi^{\rm flat,(n)}(x) = 6x(1-x)\sum_{k=0,2,\ldots}^{n} a_k^{\rm flat}\, C_k^{3/2}(2x-1)$$

- **Question:**

> what happens with the predictions of the MR model if the DAs $\phi_\pi^{\rm flat,(n)}(x)$ are used as an input ?

- **Answer:**





alternatively, check how much is contributed by each successive Gegenbauer polynomial:

$$\mathcal{F}^{MR}_{\text{flatDA}}(Q^2 = 20) \quad = \quad \underset{n=0}{3.56513 = 2.72402} + \underset{n=2}{0.648618} + \underset{n=4}{0.16226} + \underset{n=6}{0.027945} + \ldots$$

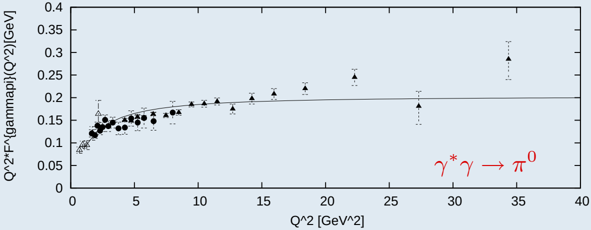
- **The moral is**

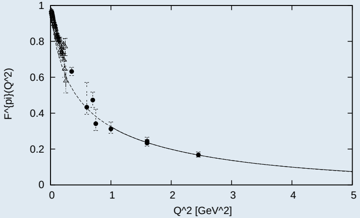
  - **The Gegenbauer expansion for the form factor calculated with flat DA converges very fast**
  - **At $Q^2 < 10 - 20$ GeV^2 using $n = 4$ truncation is sufficient**
  - **End-point behavior of a "true" pion DA is irrelevant**
  - **Radyushkin (Polyakov) describe BABAR data by introducing a large soft correction, not because of "unusual" pion DA**
  - **The "flat" values $a_2 = 0.39$ and $a_4 = 0.24$ at some low scale do not contradict the common wisdom**

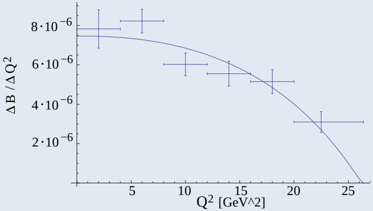
- Exact analogy: partial wave expansion

NLO LCSRs

Khodjamirian, arXiv:0909.2154







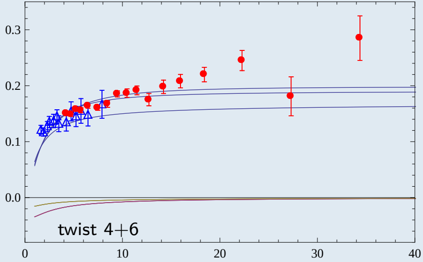
$\gamma^*\pi \to \pi$

$dB(B \to \pi e\nu)/dq^2$

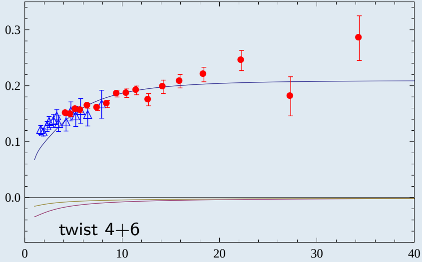
- $a_2 = 0.16, \; a_4 = 0.04$

NLO LCSRs

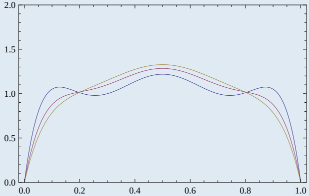
Agaev, VB, Offen, Porkert (work in progress)



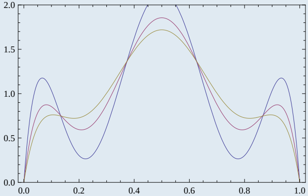
$a_2 = 0.25,\ a_4 = 0.10$



$a_2 = 0.15,\ a_4 = 0.35$

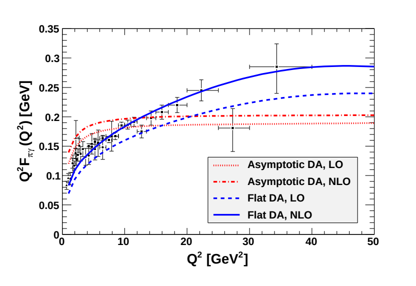


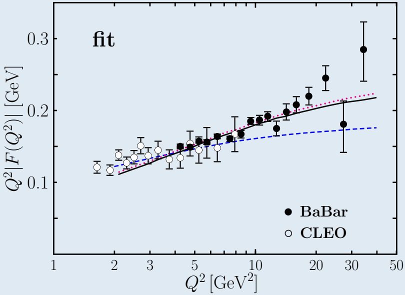
“conventional”



“exotic”

$k_\perp$ factorization





Li, Mishima, arXiv:0907.0166

- flat: $a_2 = 0.39,\ a_4 = 0.24$

P. Kroll, work in progress

- fit: $a_2 = 0.25,\ a_4 = 0.07$

Summary

- **Absence of scaling is due to large soft (Feynman) corrections**
- **Moderate scaling violation can be accomodated by theory, although it was not expected**
- **"Maximum" scaling violation (extrapolation of Babar data for higher Q^2) would create a real crisis and have considerable implications for other hard exclusive reactions, e.g. $B \to \pi \ell \nu_\ell$, DVCS etc.**

- **Using Gegenbauer expansion for pion DA with $n = 0, 2, 4$ is sufficient for CLEO and B-decays**

- **In the BaBar Q^2 range may need to add a_6 and determination of a_4 can be put as the experimental goal;**
  — presumably would need to add $n = 8 - 10$ at $Q^2 \sim 100$ GeV^2

- **More work needed on theory side**
  ↪ LCSR: generic DAs, long-distance photon, scale dependence, unbiased error analysis
  ↪ k_T factorization: constraints on acceptable WFs, NLO
  ↪ Lattice: 10% error for a_2
  ↪ $\gamma^*\gamma \to \eta, \eta'$, time-like form factors
- **More data needed in the 15–40 GeV^2 region**